ldf.1 Primitive Recursive Functions are λ-Definable

lam:ldf:prf:sec Recall that the primitive recursive functions are those that can be defined from the basic functions zero, succ, and P^n_i by composition and primitive recursion.

lam:ldf:prf:lem:basic **Lemma ldf.1.** *The basic primitive recursive functions* zero*,* succ*, and projections P^n_i are λ-definable.*

Proof. They are λ-defined by the following terms:

$$\begin{aligned}\mathrm{Zero} &\equiv \lambda a.\, \lambda f x.\, x\\ \mathrm{Succ} &\equiv \lambda a.\, \lambda f x.\, f(a f x)\\ \mathrm{Proj}^n_i &\equiv \lambda x_0 \dots x_{n-1}.\, x_i\end{aligned}$$

□

lam:ldf:prf:lem:comp **Lemma ldf.2.** *Suppose the k-ary function f, and n-ary functions $g_0, \dots, g_{k-1}$, are λ-definable by terms F, G_0, ..., G_k, and h is defined from them by composition. Then H is λ-definable*

Proof. h can be λ-defined by the term

$$H \equiv \lambda x_0 \dots x_{n-1}.\, F\,(G_0 x_0 \dots x_{n-1}) \dots (G_{k-1} x_0 \dots x_{n-1})$$

We leave verification of this fact as an exercise. □

Problem ldf.1. Complete the proof of Lemma ldf.2 by showing that $H\overline{n_0} \dots \overline{n_{n-1}} \twoheadrightarrow h(n_0, \dots, n_{n-1})$.

Note that Lemma ldf.2 did not require that f and g_0, ..., g_{k-1} are primitive recursive; it is only required that they are total and λ-definable.

lam:ldf:prf:lem:prim **Lemma ldf.3.** *Suppose f is an n-ary function and g is an $n+2$-ary function, they are λ-definable by terms F and G, and the function h is defined from f and g by primitive recursion. Then h is also λ-definable.*

Proof. Recall that h is defined by

$$\begin{aligned}h(x_1, \dots, x_n, 0) &= f(x_1, \dots, x_n)\\ h(x_1, \dots, x_n, y+1) &= h(x_1, \dots, x_n, y, h(x_1, \dots, x_n, y)).\end{aligned}$$

Informally speaking, the primitive recursive definition iterates the application of the function h y times and applies it to $f(x_1, \dots, x_n)$. This is reminiscent of the definition of Church numerals, which is also defined as a iterator.

For simplicity, we give the definition and proof for a single additional argument x. The function h is λ-defined by:

$$H \equiv \lambda x.\, \lambda y.\, \mathrm{Snd}(y D \langle \overline{0}, F x \rangle)$$

where

$$D \equiv \lambda p.\, \langle \mathrm{Succ}(\mathrm{Fst}\, p), (G x (\mathrm{Fst}\, p)(\mathrm{Snd}\, p)) \rangle$$

The iteration state we maintain is a pair, the first of which is the current y and the second is the corresponding value of h. For every step of iteration we create a pair of new values of y and h; after the iteration is done we return the second part of the pair and that's the final h value. We now prove this is indeed a representation of primitive recursion.

We want to prove that for any n and m, $H\,\overline{n}\,\overline{m} \twoheadrightarrow \overline{h(n,m)}$. To do this we first show that if $D_n \equiv D[\overline{n}/x]$, then $D_n^m\langle\overline{0}, F\,\overline{n}\rangle \twoheadrightarrow \langle\overline{m}, \overline{h(n,m)}\rangle$ We proceed by induction on m.

If $m = 0$, we want $D_n^0\langle\overline{0}, F\,\overline{n}\rangle \twoheadrightarrow \langle\overline{0}, \overline{h(n,0)}\rangle$. But $D_n^0\langle\overline{0}, F\,\overline{n}\rangle$ just is $\langle\overline{0}, F\,\overline{n}\rangle$. Since F λ-defines f, this reduces to $\langle\overline{0}, \overline{f(n)}\rangle$, and since $f(n) = h(n,0)$, this is $\langle\overline{0}, \overline{h(n,0)}\rangle$

Now suppose that $D_n^m\langle\overline{0}, F\,\overline{n}\rangle \twoheadrightarrow \langle\overline{m}, \overline{h(n,m)}\rangle$. We want to show that $D_n^{m+1}\langle\overline{0}, F\,\overline{n}\rangle \twoheadrightarrow \langle\overline{m+1}, \overline{h(n,m+1)}\rangle$.

$$
\begin{aligned}
D_n^{m+1}\langle\overline{0}, F\,\overline{n}\rangle &\equiv D_n(D_n^m\langle\overline{0}, F\,\overline{n}\rangle) \\
&\twoheadrightarrow D_n\,\langle\overline{m}, \overline{h(n,m)}\rangle \qquad \text{(by IH)} \\
&\equiv (\lambda p.\,\langle \mathrm{Succ}(\mathrm{Fst}\,p), (G\,\overline{n}(\mathrm{Fst}\,p)(\mathrm{Snd}\,p))\rangle)\langle\overline{m}, \overline{h(n,m)}\rangle \\
&\to \langle \mathrm{Succ}(\mathrm{Fst}\,\langle\overline{m}, \overline{h(n,m)}\rangle), \\
&\qquad (G\,\overline{n}(\mathrm{Fst}\,\langle\overline{m}, \overline{h(n,m)}\rangle)(\mathrm{Snd}\,\langle\overline{m}, \overline{h(n,m)}\rangle)))\rangle \\
&\twoheadrightarrow \langle \mathrm{Succ}\,\overline{m}, (G\,\overline{n}\,\overline{m}\,\overline{h(n,m)})\rangle \\
&\twoheadrightarrow \langle\overline{m+1}, \overline{g(n,m,h(n,m))}\rangle
\end{aligned}
$$

Since $g(n,m,h(n,m)) = h(n,m+1)$, we are done.

Finally, consider

$$
\begin{aligned}
H\,\overline{n}\,\overline{m} \equiv\ & \lambda x.\,\lambda y.\,\mathrm{Snd}(y(\lambda p.\langle \mathrm{Succ}(\mathrm{Fst}\,p), (G\,x\,(\mathrm{Fst}\,p)\,(\mathrm{Snd}\,p))\rangle)\langle\overline{0}, F x\rangle) \\
& \overline{n}\,\overline{m} \\
&\twoheadrightarrow \mathrm{Snd}(\overline{m}\ \underbrace{(\lambda p.\langle \mathrm{Succ}(\mathrm{Fst}\,p), (G\,\overline{n}\,(\mathrm{Fst}\,p)(\mathrm{Snd}\,p))\rangle)}_{D_n}\langle\overline{0}, F\overline{n}\rangle) \\
&\equiv \mathrm{Snd}(\overline{m}\,D_n\,\langle\overline{0}, F\overline{n}\rangle) \\
&\twoheadrightarrow \mathrm{Snd}\,(D_n^m\langle\overline{0}, F\overline{n}\rangle) \\
&\twoheadrightarrow \mathrm{Snd}\,\langle\overline{m}, \overline{h(n,m)}\rangle \\
&\twoheadrightarrow \overline{h(n,m)}.
\end{aligned}
$$

□

Proposition ldf.4. *Every primitive recursive function is λ-definable.*

Proof. By Lemma ldf.1, all basic functions are λ-definable, and by Lemma ldf.2 and Lemma ldf.3, the λ-definable functions are closed under composition and primitive recursion. □

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Bibliography