ldf.1  Minimization

The general recursive functions are those that can be obtained from the basic functions zero, succ, $P^n$ by composition, primitive recursion, and regular minimization. To show that all general recursive functions are $\lambda$-definable we have to show that any function defined by regular minimization from a $\lambda$-definable function is itself $\lambda$-definable.

**Lemma ldf.1.** If $f(x_1, \ldots, x_k, y)$ is regular and $\lambda$-definable, then $g$ defined by

$$g(x_1, \ldots, x_k) = \mu y f(x_1, \ldots, x_k, y) = 0$$

is also $\lambda$-definable.

**Proof.** Suppose the lambda term $F$ $\lambda$-defines the regular function $f(\overline{x}, y)$. To $\lambda$-define $h$ we use a search function and a fixpoint combinator:

$$\text{Search} \equiv \lambda g. \lambda f \overline{x} y. \text{IsZero}(f \overline{x} y) y (g \overline{x} (\text{Succ } y))$$

$$H \equiv \lambda\overline{x}. (\text{Y Search}) F \overline{x} \overline{0},$$

where $Y$ is any fixpoint combinator. Informally speaking, Search is a self-referencing function: starting with $y$, test whether $f \overline{x} y$ is zero: if so, return $y$, otherwise call itself with Succ $y$. Thus $(\text{Y Search}) F \overline{n_1} \ldots \overline{n_k} \overline{0}$ returns the least $m$ for which $f(n_1, \ldots, n_k, m) = 0$.

Specifically, observe that

$$(\text{Y Search}) F \overline{n_1} \ldots \overline{n_k} m \leftrightarrow m$$

if $f(n_1, \ldots, n_k, m) = 0$, or

$$\leftrightarrow (\text{Y Search}) F \overline{n_1} \ldots \overline{n_k} m + 1$$

otherwise. Since $f$ is regular, $f(n_1, \ldots, n_k, y) = 0$ for some $y$, and so

$$(\text{Y Search}) F \overline{n_1} \ldots \overline{n_k} \overline{0} \leftrightarrow h(n_1, \ldots, n_k).$$

**Proposition ldf.2.** Every general recursive function is $\lambda$-definable.

**Proof.** By ??, all basic functions are $\lambda$-definable, and by ??, ??, and Lemma ldf.1, the $\lambda$-definable functions are closed under composition, primitive recursion, and regular minimization.

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Bibliography