Minimization

The general recursive functions are those that can be obtained from the basic functions zero, succ, \( P^n \) by composition, primitive recursion, and regular minimization. To show that all general recursive functions are \( \lambda \)-definable we have to show that any function defined by regular minimization from a \( \lambda \)-definable function is itself \( \lambda \)-definable.

Lemma ldf.1. If \( f(x_1, \ldots, x_k, y) \) is regular and \( \lambda \)-definable, then \( g \) defined by

\[
g(x_1, \ldots, x_k) = \mu y \ f(x_1, \ldots, x_k, y) = 0
\]
is also \( \lambda \)-definable.

Proof. Suppose the lambda term \( F \) \( \lambda \)-defines the regular function \( f(\bar{x}, y) \). To \( \lambda \)-define \( h \) we use a search function and a fixpoint combinator:

\[
\text{Search} ≡ λg. \ λf \bar{x} y. \ \text{IsZero}(f \bar{x} y) \ y (g \bar{x}(\text{Succ} y))
\]

\[
H ≡ λ\bar{x}. \ (Y \text{Search})F \bar{x} 0,
\]

where \( Y \) is any fixpoint combinator. Informally speaking, Search is a self-referencing function: starting with \( y \), test whether \( f \bar{x} y \) is zero: if so, return \( y \), otherwise call itself with \( \text{Succ} y \). Thus \( (Y \text{Search})F \bar{n}_1 \ldots \bar{n}_k 0 \) returns the least \( m \) for which \( f(n_1, \ldots, n_k, m) = 0 \).

Specifically, observe that

\[
(Y \text{Search})F \bar{n}_1 \ldots \bar{n}_k m \twoheadrightarrow m
\]

if \( f(n_1, \ldots, n_k, m) = 0 \), or

\[
\twoheadrightarrow (Y \text{Search})F \bar{n}_1 \ldots \bar{n}_k m + 1
\]

otherwise. Since \( f \) is regular, \( f(n_1, \ldots, n_k, y) = 0 \) for some \( y \), and so

\[
(Y \text{Search})F \bar{n}_1 \ldots \bar{n}_k 0 \twoheadrightarrow h(n_1, \ldots, n_k). \quad \Box
\]

Proposition ldf.2. Every general recursive function is \( \lambda \)-definable.

Proof. By ??, all basic functions are \( \lambda \)-definable, and by ??, ??, and Lemma ldf.1, the \( \lambda \)-definable functions are closed under composition, primitive recursion, and regular minimization. \( \Box \)

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Bibliography