

int.1 Reduction of Lambda Terms

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sec

What can one do with lambda terms? Simplify them. If M and N are any lambda terms and x is any variable, we can use $M[N/x]$ to denote the result of substituting N for x in M , after renaming any bound variables of M that would interfere with the free variables of N after the substitution. For example,

$$(\lambda w. xxw)[yyz/x] = \lambda w. (yyz)(yyz)w.$$

Alternative notations for substitution are $[N/x]M$, $M[N/x]$, and also $M[x/N]$ ^{digression}. Beware!

Intuitively, $(\lambda x. M)N$ and $M[N/x]$ have the same meaning; the act of replacing the first term by the second is called β -contraction. $(\lambda x. M)N$ is called a *redex* and $M[N/x]$ its *contractum*. Generally, if it is possible to change a term P to P' by β -contraction of some subterm, we say that P β -reduces to P' in one step, and write $P \rightarrow P'$. If from P we can obtain P' with some number of one-step reductions (possibly none), then P β -reduces to P' ; in symbols, $P \twoheadrightarrow P'$. A term that cannot be β -reduced any further is called β -irreducible, or β -normal. We will say “reduces” instead of “ β -reduces,” etc., when the context is clear.

Let us consider some examples.

1. We have

$$\begin{aligned}(\lambda x. xxy)\lambda z. z &\rightarrow (\lambda z. z)(\lambda z. z)y \\ &\rightarrow (\lambda z. z)y \\ &\rightarrow y\end{aligned}$$

2. “Simplifying” a term can make it more complex:

$$\begin{aligned}(\lambda x. xxy)(\lambda x. xxy) &\rightarrow (\lambda x. xxy)(\lambda x. xxy)y \\ &\rightarrow (\lambda x. xxy)(\lambda x. xxy)yy \\ &\rightarrow \dots\end{aligned}$$

3. It can also leave a term unchanged:

$$(\lambda x. xx)(\lambda x. xx) \rightarrow (\lambda x. xx)(\lambda x. xx)$$

4. Also, some terms can be reduced in more than one way; for example,

$$(\lambda x. (\lambda y. yx)z)v \rightarrow (\lambda y. yv)z$$

by contracting the outermost application; and

$$(\lambda x. (\lambda y. yx)z)v \rightarrow (\lambda x. zx)v$$

by contracting the innermost one. Note, in this case, however, that both terms further reduce to the same term, zv .

The final outcome in the last example is not a coincidence, but rather illustrates a deep and important property of the lambda calculus, known as the “Church-Rosser property.”

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Bibliography