int.1 Reduction of Lambda Terms

What can one do with lambda terms? Simplify them. If $M$ and $N$ are any lambda terms and $x$ is any variable, we can use $M[N/x]$ to denote the result of substituting $N$ for $x$ in $M$, after renaming any bound variables of $M$ that would interfere with the free variables of $N$ after the substitution. For example,

$$(\lambda w. xxw)[yyz/x] = \lambda w. (yyz)(yyz)w.$$ 

Alternative notations for substitution are $[N/x]M$, $[x/N]M$, and also $M[x/N]$. Beware!

Intuitively, $(\lambda x. M)N$ and $M[N/x]$ have the same meaning: the act of replacing the first term by the second is called $\beta$-contraction. $(\lambda x. M)N$ is called a redex and $M[N/x]$ its contractum. Generally, if it is possible to change a term $P$ to $P'$ by $\beta$-contraction of some subterm, we say that $P$ $\beta$-reduces to $P'$ in one step, and write $P \rightarrow P'$. If from $P$ we can obtain $P'$ with some number of one-step reductions (possibly none), then $P$ $\beta$-reduces to $P'$; in symbols, $P \rightarrow \redex P'$. A term that cannot be $\beta$-reduced any further is called $\beta$-irreducible, or $\beta$-normal. We will say “reduces” instead of “$\beta$-reduces,” etc., when the context is clear.

Let us consider some examples.

1. We have

$$(\lambda x. xxy)\lambda z. z \rightarrow (\lambda z. z)(\lambda z. z)y$$
$$\rightarrow (\lambda z. z)y$$
$$\rightarrow y.$$ 

2. “Simplifying” a term can make it more complex:

$$(\lambda x. xxy)(\lambda x. xxy) \rightarrow (\lambda x. xxy)(\lambda x. xxy)y$$
$$\rightarrow (\lambda x. xxy)(\lambda x. xxy)yy$$
$$\rightarrow \ldots.$$ 

3. It can also leave a term unchanged:

$$(\lambda x. xx)(\lambda x. xx) \rightarrow (\lambda x. xx)(\lambda x. xx).$$ 

4. Also, some terms can be reduced in more than one way; for example,

$$(\lambda x. (\lambda y. yx)z)v \rightarrow (\lambda y. yv)z$$

by contracting the outermost application; and

$$(\lambda x. (\lambda y. yx)z)v \rightarrow (\lambda x. zv)$$

by contracting the innermost one. Note, in this case, however, that both terms further reduce to the same term, $zv$. 

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The final outcome in the last example is not a coincidence, but rather illustrates a deep and important property of the lambda calculus, known as the “Church-Rosser property.”

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Bibliography