int.1 Reduction of Lambda Terms

lam:int:red:sec

What can one do with lambda terms? Simplify them. If M and N are any lambda terms and x is any variable, we can use $M[N/x]$ to denote the result of substituting N for x in M, after renaming any bound variables of M that would interfere with the free variables of N after the substitution. For example,

$$(\lambda w.\, xxw)[yyz/x] = \lambda w.\, (yyz)(yyz)w.$$

Alternative notations for substitution are $[N/x]M$, $[x/N]M$, and also $M[x/N]$. Beware! digression

Intuitively, $(\lambda x.\, M)N$ and $M[N/x]$ have the same meaning; the act of replacing the first term by the second is called *β-contraction*. $(\lambda x.\, M)N$ is called a *redex* and $M[N/x]$ its *contractum*. Generally, if it is possible to change a term P to P' by β-contraction of some subterm, we say that *P β-reduces to P' in one step*, and write $P \to P'$. If from P we can obtain P' with some number of one-step reductions (possibly none), then P *β-reduces* to P'; in symbols, $P \twoheadrightarrow P'$. A term that cannot be β-reduced any further is called *β-irreducible*, or *β-normal*. We will say "reduces" instead of "β-reduces," etc., when the context is clear.

Let us consider some examples.

1. We have

$$\begin{aligned}(\lambda x.\, xxy)\lambda z.\, z &\to (\lambda z.\, z)(\lambda z.\, z)y\\ &\to (\lambda z.\, z)y\\ &\to y.\end{aligned}$$

2. "Simplifying" a term can make it more complex:

$$\begin{aligned}(\lambda x.\, xxy)(\lambda x.\, xxy) &\to (\lambda x.\, xxy)(\lambda x.\, xxy)y\\ &\to (\lambda x.\, xxy)(\lambda x.\, xxy)yy\\ &\to \dots\end{aligned}$$

3. It can also leave a term unchanged:

$$(\lambda x.\, xx)(\lambda x.\, xx) \to (\lambda x.\, xx)(\lambda x.\, xx).$$

4. Also, some terms can be reduced in more than one way; for example,

$$(\lambda x.\, (\lambda y.\, yx)z)v \to (\lambda y.\, yv)z$$

by contracting the outermost application; and

$$(\lambda x.\, (\lambda y.\, yx)z)v \to (\lambda x.\, zx)v$$

by contracting the innermost one. Note, in this case, however, that both terms further reduce to the same term, zv.

The final outcome in the last example is not a coincidence, but rather illustrates a deep and important property of the lambda calculus, known as the "Church–Rosser property."

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Bibliography