Lemma \textit{int.1}. Suppose \( f(x, y) \) is primitive recursive. Let \( g \) be defined by

\[ g(x) \simeq \mu y \ f(x, y). \]

Then \( g \) is \( \lambda \)-definable.

\textit{Proof.} The idea is roughly as follows. Given \( x \), we will use the fixed-point lambda term \( Y \) to define a function \( h_x(n) \) which searches for a \( y \) starting at \( n \); then \( g(x) \) is just \( h_x(0) \). The function \( h_x \) can be expressed as the solution of a fixed-point equation:

\[ h_x(n) \simeq \begin{cases} n & \text{if } f(x, n) = 0 \\ h_x(n+1) & \text{otherwise}. \end{cases} \]

Here are the details. Since \( f \) is primitive recursive, it is \( \lambda \)-defined by some term \( F \). Remember that we also have a lambda term \( D \), such that \( D(M, N, \bar{0}) \rightarrow M \) and \( D(M, N, \bar{1}) \rightarrow N \). Fixing \( x \) for the moment, to \( \lambda \)-define \( h_x \) we want to find a term \( H \) (depending on \( x \)) satisfying

\[ H(\bar{n}) \equiv D(\bar{n}, H(S(\bar{n})), F(x, \bar{n})). \]

We can do this using the fixed-point term \( Y \). First, let \( U \) be the term

\[ \lambda h. \lambda z. D(z, (h(Sz)), F(x, z)), \]

and then let \( H \) be the term \( YU \). Notice that the only free variable in \( H \) is \( x \). Let us show that \( H \) satisfies the equation above.

By the definition of \( Y \), we have

\[ H = YU \equiv U(YU) = U(H). \]

In particular, for each natural number \( n \), we have

\[ H(\bar{n}) \equiv U(H, \bar{n}) \]

\[ \rightarrow D(\bar{n}, H(S(\bar{n})), F(x, \bar{n})), \]

as required. Notice that if you substitute a numeral \( \bar{m} \) for \( x \) in the last line, the expression reduces to \( \bar{n} \) if \( F(\bar{m}, \bar{n}) \) reduces to \( \bar{0} \), and it reduces to \( H(S(\bar{n})) \) if \( F(\bar{m}, \bar{n}) \) reduces to any other numeral.

To finish off the proof, let \( G \) be \( \lambda x. \ H(\bar{0}) \). Then \( G \) \( \lambda \)-defines \( g \); in other words, for every \( m \), \( G(\bar{m}) \) reduces to \( g(m) \), if \( g(m) \) is defined, and has no normal form otherwise. \( \square \)
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Bibliography