

int.1 The λ -Definable Functions are Closed under Minimization

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sec

Lemma int.1. *Suppose $f(x, y)$ is primitive recursive. Let g be defined by*

$$g(x) \simeq \mu y f(x, y).$$

Then g is λ -definable.

Proof. The idea is roughly as follows. Given x , we will use the fixed-point lambda term Y to define a function $h_x(n)$ which searches for a y starting at n ; then $g(x)$ is just $h_x(0)$. The function h_x can be expressed as the solution of a fixed-point equation:

$$h_x(n) \simeq \begin{cases} n & \text{if } f(x, n) = 0 \\ h_x(n+1) & \text{otherwise.} \end{cases}$$

Here are the details. Since f is primitive recursive, it is λ -defined by some term F . Remember that we also have a lambda term D , such that $D(M, N, \bar{0}) \rightarrow M$ and $D(M, N, \bar{1}) \rightarrow N$. Fixing x for the moment, to λ -define h_x we want to find a term H (depending on x) satisfying

$$H(\bar{n}) \equiv D(\bar{n}, H(S\bar{n}), F(x, \bar{n})).$$

We can do this using the fixed-point term Y . First, let U be the term

$$\lambda h. \lambda z. D(z, (h(Sz)), F(x, z)),$$

and then let H be the term YU . Notice that the only free variable in H is x . Let us show that H satisfies the equation above.

By the definition of Y , we have

$$H = YU \equiv U(YU) = U(H).$$

In particular, for each natural number n , we have

$$\begin{aligned} H(\bar{n}) &\equiv U(H, \bar{n}) \\ &\rightarrow D(\bar{n}, H(S\bar{n}), F(x, \bar{n})), \end{aligned}$$

as required. Notice that if you substitute a numeral \bar{m} for x in the last line, the expression reduces to \bar{n} if $F(\bar{m}, \bar{n})$ reduces to $\bar{0}$, and it reduces to $H(S\bar{n})$ if $F(\bar{m}, \bar{n})$ reduces to any other numeral.

To finish off the proof, let G be $\lambda x. H(\bar{0})$. Then G λ -defines g ; in other words, for every m , $G(\bar{m})$ reduces to $\bar{g(m)}$, if $g(m)$ is defined, and has no normal form otherwise. \square

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Bibliography