int.1 λ-Definable Arithmetical Functions

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How can the lambda calculus serve as a model of computation? At first, it is not even clear how to make sense of this statement. To talk about computability on the natural numbers, we need to find a suitable representation for such numbers. Here is one that works surprisingly well.

Definition int.1. For each natural number n, define the *Church numeral* $\overline{n}$ to be the lambda term $\lambda x.\, \lambda y.\, (x(x(x(\ldots x(y)))))$, where there are n x's in all.

The terms $\overline{n}$ are "iterators": on input f, $\overline{n}$ returns the function mapping y to $f^n(y)$. Note that each numeral is normal. We can now say what it means for a lambda term to "compute" a function on the natural numbers.

Definition int.2. Let $f(x_0, \ldots, x_{k-1})$ be an n-ary partial function from $\mathbb{N}$ to $\mathbb{N}$. We say a λ-term F *λ-defines* f iff for every sequence of natural numbers $n_0, \ldots, n_{k-1}$,

$$F\,\overline{n_0}\,\overline{n_1}\ldots\overline{n_{k-1}} \twoheadrightarrow \overline{f(n_0, n_1, \ldots, n_{k-1})}$$

if $f(n_0, \ldots, n_{k-1})$ is defined, and $F, \overline{n_0}\,\overline{n_1}\ldots\overline{n_{k-1}}$ has no normal form otherwise.

lam:int:rep:thm:lambda-def

Theorem int.3. *A function f is a partial computable function if and only if it is λ-defined by a lambda term.*

This theorem is somewhat striking. As a model of computation, the lambda calculus is a rather simple calculus; the only operations are lambda abstraction and application! From these meager resources, however, it is possible to implement any computational procedure. explanation

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Bibliography