int.1 Fixed-Point Combinators

lam:int:fix:sec Suppose you have a lambda term g, and you want another term k with the property that k is β-equivalent to gk. Define terms

$$\mathrm{diag}(x) = xx$$

and

$$l(x) = g(\mathrm{diag}(x))$$

using our notational conventions; in other words, l is the term $\lambda x.\, g(xx)$. Let k be the term ll. Then we have

$$\begin{aligned} k &= (\lambda x.\, g(xx))(\lambda x.\, g(xx)) \\ &\twoheadrightarrow g((\lambda x.\, g(xx))(\lambda x.\, g(xx))) \\ &= gk. \end{aligned}$$

If one takes

$$Y = \lambda g.\, ((\lambda x.\, g(xx))(\lambda x.\, g(xx)))$$

then $Y g$ and $g(Y g)$ reduce to a common term; so $Y g \equiv_\beta g(Y g)$. This is known as "Curry's combinator." If instead one takes

$$Y = (\lambda xg.\, g(xxg))(\lambda xg.\, g(xxg))$$

then in fact $Y g$ reduces to $g(Y g)$, which is a stronger statement. This latter version of Y is known as "Turing's combinator."

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Bibliography