int.1  Fixed-Point Combinators

Suppose you have a lambda term \(g\), and you want another term \(k\) with the property that \(k\) is \(\beta\)-equivalent to \(gk\). Define terms
\[
diag(x) = xx
\]
and
\[
l(x) = g(diag(x))
\]
using our notational conventions; in other words, \(l\) is the term \(\lambda x. g(xx)\). Let \(k\) be the term \(ll\). Then we have
\[
k = (\lambda x. g(xx))(\lambda x. g(xx))
\rightarrow g((\lambda x. g(xx))(\lambda x. g(xx)))
= gk.
\]
If one takes
\[
Y = \lambda g. ((\lambda x. g(xx))(\lambda x. g(xx)))
\]
then \(Yg\) and \(g(Yg)\) reduce to a common term; so \(Yg \equiv \beta g(Yg)\). This is known as “Curry’s combinator.” If instead one takes
\[
Y = (\lambda xg. g(xxg))(\lambda xg. g(xxg))
\]
then in fact \(Yg\) reduces to \(g(Yg)\), which is a stronger statement. This latter version of \(Y\) is known as “Turing’s combinator.”

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Bibliography