Theorem int.1. Every computable partial function is \(\lambda\)-definable.

Proof. We need to show that every partial computable function \(f\) is \(\lambda\)-defined by a lambda term \(F\). By Kleene’s normal form theorem, it suffices to show that every primitive recursive function is \(\lambda\)-defined by a lambda term, and then that the functions \(\lambda\)-definable are closed under suitable compositions and unbounded search. To show that every primitive recursive function is \(\lambda\)-defined by a lambda term, it suffices to show that the initial functions are \(\lambda\)-definable, and that the partial functions that are \(\lambda\)-definable are closed under composition, primitive recursion, and unbounded search.

We will use a more conventional notation to make the rest of the proof more readable. For example, we will write \(M(x, y, z)\) instead of \(Mxyz\). While this is suggestive, you should remember that terms in the untyped lambda calculus do not have associated arities; so, for the same term \(M\), it makes just as much sense to write \(M(x, y)\) and \(M(x, y, z, w)\). But using this notation indicates that we are treating \(M\) as a function of three variables, and helps make the intentions behind the definitions clearer. In a similar way, we will say “define \(M\) by \(M(x, y, z) = \ldots\)” instead of “define \(M\) by \(M = \lambda x. \lambda y. \lambda z. \ldots\)”.

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Bibliography