The Church-Rosser Property

**Theorem int.1.** Let $M$, $N_1$, and $N_2$ be terms, such that $M \rightarrow N_1$ and $M \rightarrow N_2$. Then there is a term $P$ such that $N_1 \rightarrow P$ and $N_2 \rightarrow P$.

**Corollary int.2.** Suppose $M$ can be reduced to normal form. Then this normal form is unique.

**Proof.** If $M \rightarrow N_1$ and $M \rightarrow N_2$, by the previous theorem there is a term $P$ such that $N_1$ and $N_2$ both reduce to $P$. If $N_1$ and $N_2$ are both in normal form, this can only happen if $N_1 \equiv P \equiv N_2$.  

Finally, we will say that two terms $M$ and $N$ are \( \beta \)-equivalent, or just equivalent, if they reduce to a common term; in other words, if there is some $P$ such that $M \rightarrow P$ and $N \rightarrow P$. This is written $M \overset{\beta}{=} N$. Using Theorem int.1, you can check that \( \overset{\beta}{=} \) is an equivalence relation, with the additional property that for every $M$ and $N$, if $M \rightarrow N$ or $N \rightarrow M$, then $M \overset{\beta}{=} N$. (In fact, one can show that \( \overset{\beta}{=} \) is the smallest equivalence relation having this property.)

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Bibliography