Parallel $\beta$-reduction

We introduce the notion of parallel $\beta$-reduction, and prove the it has the Church–Rosser property.

**Definition cr.1 (parallel $\beta$-reduction, $\rightarrow^\parallel$).** Parallel reduction ($\rightarrow^\parallel$) of terms is inductively defined as follows:

1. $x \rightarrow^\parallel x$.
2. If $N \rightarrow^\beta N'$ then $\lambda x. N \rightarrow^\parallel \lambda x. N'$.
3. If $P \rightarrow^\beta P'$ and $Q \rightarrow^\beta Q'$ then $PQ \rightarrow^\parallel P'Q'$.
4. If $N \rightarrow^\beta N'$ and $Q \rightarrow^\beta Q'$ then $(\lambda x. N)Q \rightarrow^\parallel N'[Q'/x]$.

Parallel $\beta$-reduction allows us to reduce any number of redices in a term in one step. It is different from $\beta$-reduction in the sense that we can only contract redices that occur in the original term, but not redices arising from parallel $\beta$-reduction. For example, the term $(\lambda f. fx)(\lambda y. y)$ can only be parallel $\beta$-reduced to itself or to $(\lambda y. y)x$, but not further to $x$, although it $\beta$-reduces to $x$, because this redex arises only after one step of parallel $\beta$-reduction. A second parallel $\beta$-reduction step yields $x$, though.

**Theorem cr.2.** $M \rightarrow^\parallel M$.

**Proof.** Exercise.

**Problem cr.1.** Prove Theorem cr.2.

**Definition cr.3 ($\beta$-complete development).** The $\beta$-complete development $M^{*\beta}$ of $M$ is defined inductively as follows:

$$x^{*\beta} = x$$

$$\lambda x. N^{*\beta} = \lambda x. N^{*\beta}$$

$$PQ^{*\beta} = P^{*\beta}Q^{*\beta}$$

if $P$ is not a $\lambda$-abstract

$$((\lambda x. N)Q)^{*\beta} = N^{*\beta}[Q^{*\beta}/x]$$

The $\beta$-complete development of a term, as its name suggests, is a “complete parallel reduction.” While for parallel $\beta$-reduction we still can choose to not contract a redex, for complete development we have no choice but to contract all of them. Thus the complete development of $(\lambda f. fx)(\lambda y. y)$ is $(\lambda y. y)x$, not itself.
This definition has the problem that we haven’t introduced how to define functions on ($\lambda$-)terms recursively. Will fix in future.

**Lemma cr.4.** If $M \xrightarrow{\beta} M'$ and $R \xrightarrow{\beta} R'$, then $M[R/y] \xrightarrow{\beta} M'[R'/y]$.

*Proof.* By induction on the derivation of $M \xrightarrow{\beta} M'$.

1. The last step is (1): Exercise.

2. The last step is (2): Then $M$ is $\lambda x. N$ and $M'$ is $\lambda x. N'$, where $N \xrightarrow{\beta} N'$. We want to prove that $(\lambda x. N)[R/y] \xrightarrow{\beta} (\lambda x. N')[R'/y]$, i.e., $\lambda x. N[R/y] \xrightarrow{\beta} \lambda x. N'[R'/y]$. This follows immediately by (2) and the induction hypothesis.

3. The last step is (3): Exercise.

4. The last step is (4): $M$ is $(\lambda x. N)Q$ and $M'$ is $N'[Q'/x]$. We want to prove that $((\lambda x. N)Q)[R/y] \xrightarrow{\beta} N'[Q'/x][R'/y]$, i.e., $(\lambda x. N[R/y])Q[R/y] \xrightarrow{\beta} N'[R'/y][Q'[R'/y]/x]$. This follows by (4) and the induction hypothesis.

**Problem cr.2.** Complete the proof of Lemma cr.4.

**Lemma cr.5.** If $M \xrightarrow{\beta} M'$ then $M' \xrightarrow{\beta} M^{*\beta}$.

*Proof.* By induction on the derivation of $M \xrightarrow{\beta} M'$.

1. The last rule is (1): Exercise.

2. The last rule is (2): $M$ is $\lambda x. N$ and $M'$ is $\lambda x. N'$ with $N \xrightarrow{\beta} N'$. We want to show that $\lambda x. N' \xrightarrow{\beta} (\lambda x. N)^{*\beta}$, i.e., $\lambda x. N' \xrightarrow{\beta} \lambda x. N^{*\beta}$ by eq. (2). It follows by (2) and the induction hypothesis.

3. The last rule is (3): $M$ is $PQ$ and $M'$ is $P'Q'$ for some $P$, $Q$, $P'$ and $Q'$, with $P \xrightarrow{\beta} P'$ and $Q \xrightarrow{\beta} Q'$. By induction hypothesis, we have $P' \xrightarrow{\beta} P^{*\beta}$ and $Q' \xrightarrow{\beta} Q^{*\beta}$.

   a) If $P$ is $\lambda x. N$ for some $x$ and $N$, then $P'$ must be $\lambda x. N'$ for some $N'$ with $N \xrightarrow{\beta} N'$. By induction hypothesis we have $N' \xrightarrow{\beta} N^{*\beta}$ and $Q' \xrightarrow{\beta} Q^{*\beta}$. Then $(\lambda x. N')Q' \xrightarrow{\beta} N^{*\beta}[Q^{*\beta}/x]$ by (4).

   b) If $P$ is not a $\lambda$-abstract, then $P'Q' \xrightarrow{\beta} P^{*\beta}Q^{*\beta}$ by (3), and the right-hand side is $PQ^{*\beta}$ by eq. (3).
4. The last rule is (4): $M$ is $(\lambda x. N)Q$ and $M'$ is $N'[Q'/x]$ for some $x$, $N$, $Q$, $N'$, and $Q'$, with $N \xrightarrow{\beta} N'$ and $Q \xrightarrow{\beta} Q'$. By induction hypothesis we know $N' \xrightarrow{\beta} N^* \beta$ and $Q' \xrightarrow{\beta} Q^* \beta$. By Lemma cr.4 we have $N'[Q'/x] \xrightarrow{\beta} N^* \beta [Q^* \beta / x]$, the right-hand side of which is exactly $((\lambda x. N)Q)^* \beta$. □

Problem cr.3. Complete the proof of Lemma cr.5.

Theorem cr.6. $\xrightarrow{\beta}$ has the Church–Rosser property.  

\textit{Proof.} Immediate from Lemma cr.5. □

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Bibliography