We introduce the notion of parallel $\beta$-reduction, and prove the it has the Church-Rosser property.

**Definition cr.1 (parallel $\beta$-reduction, $\parallel\beta\Rightarrow$).** Parallel reduction ($\parallel\beta\Rightarrow$) of terms is inductively defined as follows:

1. $x \parallel\beta\Rightarrow x$.
2. If $N \beta\Rightarrow N'$ then $\lambda x. N \parallel\beta\Rightarrow \lambda x. N'$.
3. If $P \parallel\beta\Rightarrow P'$ and $Q \parallel\beta\Rightarrow Q'$ then $PQ \parallel\beta\Rightarrow P'Q'$.
4. If $N \parallel\beta\Rightarrow N'$ and $Q \parallel\beta\Rightarrow Q'$ then $(\lambda x. N)Q \parallel\beta\Rightarrow N'[Q'/x]$.

Parallel $\beta$-reduction allows us to reduce any number of redices in a term in one step. It is different from $\beta$-reduction in the sense that we can only contract redices that occur in the original term, but not redices arising from parallel $\beta$-reduction. For example, the term $(\lambda f. fx)(\lambda y. y)$ can only be parallel $\beta$-reduced to itself or to $(\lambda y. y)x$, but not further to $x$, although it $\beta$-reduces to $x$, because this redex arises only after one step of parallel $\beta$-reduction. A second parallel $\beta$-reduction step yields $x$, though.

**Theorem cr.2.** $M \parallel\beta\Rightarrow M$.

*Proof.* Exercise. $\square$

**Problem cr.1.** Prove Theorem cr.2.

**Definition cr.3 ($\beta$-complete development).** The $\beta$-complete development $M^{\beta}$ of $M$ is defined inductively as follows:

1. $x^{\beta} = x$  \hspace{1cm} (1)
2. $(\lambda x. N)^{\beta} = \lambda x. N^{\beta}$  \hspace{1cm} (2)
3. $(PQ)^{\beta} = P^{\beta}Q^{\beta}$ if $P$ is not a $\lambda$-abstract  \hspace{1cm} (3)
4. $((\lambda x. N)Q)^{\beta} = N^{\beta}[Q^{\beta}/x]$  \hspace{1cm} (4)

The $\beta$-complete development of a term, as its name suggests, is a “complete parallel reduction.” While for parallel $\beta$-reduction we still can choose to not contract a redex, for complete development we have no choice but to contract all of them. Thus the complete development of $(\lambda f. fx)(\lambda y. y)$ is $(\lambda y. y)x$, not itself.
This definition has the problem that we haven’t introduced how to define functions on (\(\lambda\)-)terms recursively. Will fix in future.

**Lemma cr.4.** If \(M \xrightarrow{\beta} M'\) and \(R \xrightarrow{\beta} R'\), then \(M[R/y] \xrightarrow{\beta} M'[R'/y]\).

**Proof.** By induction on the derivation of \(M \xrightarrow{\beta} M'\).

1. The last step is (1): Exercise.

2. The last step is (2): Then \(M = \lambda x. N\) and \(M' = \lambda x. N'\), where \(N \xrightarrow{\beta} N'\). We want to prove that \((\lambda x. N)[R/y] \xrightarrow{\beta} (\lambda x. N')[[R'/y], i.e., \(\lambda x. N[R/y] \xrightarrow{\beta} \lambda x. N'[R'/y]\). This follows immediately by (2) and the induction hypothesis.

3. The last step is (3): Exercise.

4. The last step is (4): \(M = (\lambda x. N)Q\) and \(M' = N'[Q'/x]\). We want to prove that \(((\lambda x. N)Q)[R/y] \xrightarrow{\beta} N'[Q'/x][R'/y]\), i.e., \((\lambda x. N[R/y])Q[R/y] \xrightarrow{\beta} N'[R'/y][Q'[R'/y]/x]\). This follows by (4) and the induction hypothesis.

**Problem cr.2.** Complete the proof of Lemma cr.4.

**Lemma cr.5.** If \(M \xrightarrow{\beta} M'\) then \(M' \xrightarrow{\beta} M^{*\beta}\).

**Proof.** By induction on the derivation of \(M \xrightarrow{\beta} M'\).

1. The last rule is (1): Exercise.

2. The last rule is (2): \(M = \lambda x. N\) and \(M' = \lambda x. N'\) with \(N \xrightarrow{\beta} N'\). We want to show that \(\lambda x. N' \xrightarrow{\beta} (\lambda x. N)^{*\beta}\), i.e., \(\lambda x. N' \xrightarrow{\beta} \lambda x. N^{*\beta}\) by eq. (2). It follows by (2) and the induction hypothesis.

3. The last rule is (3): \(M = PQ\) and \(M' = P'Q'\) for some \(P, Q, P'\) and \(Q'\), with \(P \xrightarrow{\beta} P'\) and \(Q \xrightarrow{\beta} Q'\). By induction hypothesis, we have \(P' \xrightarrow{\beta} P^{*\beta}\) and \(Q' \xrightarrow{\beta} Q^{*\beta}\).

   a) If \(P\) is \(\lambda x. N\) for some \(x\) and \(N\), then \(P'\) must be \(\lambda x. N'\) for some \(N'\) with \(N \xrightarrow{\beta} N'\). By induction hypothesis we have \(N' \xrightarrow{\beta} N'^{*\beta}\) and \(Q' \xrightarrow{\beta} Q^{*\beta}\). Then \((\lambda x. N')[Q'[R'/y] \xrightarrow{\beta} N'^{*\beta}[Q'^{*\beta}/x]\) by (4).

   b) If \(P\) is not a \(\lambda\)-abstract, then \(P'Q' \xrightarrow{\beta} P^{*\beta}Q^{*\beta}\) by (3), and the right-hand side is \(PQ^{*\beta}\) by eq. (3).

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4. The last rule is (4): $M$ is $(\lambda x. N)Q$ and $M'$ is $N'[Q'/x]$ for some $x$, $N$, $Q$, $N'$, and $Q'$, with $N \overset{\beta}{\Rightarrow} N'$ and $Q \overset{\beta}{\Rightarrow} Q'$. By induction hypothesis we know $N' \overset{\beta}{\Rightarrow} N'^{\beta}$ and $Q' \overset{\beta}{\Rightarrow} Q'^{\beta}$. By Lemma cr.4 we have $N'[Q'/x] \overset{\beta}{\Rightarrow} N'^{\beta}[Q'^{\beta}/x]$, the right-hand side of which is exactly $((\lambda x. N)Q)^{\beta}$. □

Problem cr.3. Complete the proof of Lemma cr.5.

Theorem cr.6. $\Rightarrow$ has the Church-Rosser property.

Proof. Immediate from Lemma cr.5. □

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Bibliography