**cr.1 Parallel \(\beta\)-reduction**

We introduce the notion of parallel \(\beta\)-reduction, and prove that it has the Church-Rosser property.

**Definition cr.1 (parallel \(\beta\)-reduction, \(\equiv\)).** Parallel reduction (\(\equiv\)) of terms is inductively defined as follows:

1. \(x \equiv x\).
2. If \(N \equiv N'\) then \(\lambda x. N \equiv \lambda x. N'\).
3. If \(P \equiv P'\) and \(Q \equiv Q'\) then \(PQ \equiv P'Q'\).
4. If \(N \equiv N'\) and \(Q \equiv Q'\) then \((\lambda x. N)Q \equiv N'[Q'/x]\).

Parallel \(\beta\)-reduction allows us to reduce any number of redices in a term in one step. It is different from \(\beta\)-reduction in the sense that we can only contract redices that occur in the original term, but not redices arising from parallel \(\beta\)-reduction. For example, the term \((\lambda f. fx)(\lambda y. y)\) can only be parallel \(\beta\)-reduced to itself or to \((\lambda y. y)x\), but not further to \(x\), although it \(\beta\)-reduces to \(x\), because this redex arises only after one step of parallel \(\beta\)-reduction. A second parallel \(\beta\)-reduction step yields \(x\), though.

**Theorem cr.2.** \(M \equiv M\).

*Proof. Exercise.*

**Problem cr.1.** Prove Theorem cr.2.

**Definition cr.3 (\(\beta\)-complete development).** The \(\beta\)-complete development \(M^{*\beta}\) of \(M\) is defined inductively as follows:

\[
x^{*\beta} = x \quad \text{(1)}
\]
\[
(\lambda x. N)^{\beta*} = \lambda x. N^{\beta*} \quad \text{(2)}
\]
\[
(PQ)^{\beta*} = P^{\beta*} Q^{\beta*} \quad \text{if } P \text{ is not a } \lambda\text{-abstract} \quad \text{(3)}
\]
\[
((\lambda x. N)Q)^{\beta*} = N^{\beta*}[Q^{\beta*}/x] \quad \text{(4)}
\]

The \(\beta\)-complete development of a term, as its name suggests, is a “complete parallel reduction.” While for parallel \(\beta\)-reduction we can choose to not contract a redex, for complete development we have no choice but to contract all of them. Thus the complete development of \((\lambda f. fx)(\lambda y. y)\) is \((\lambda y. y)x\), not itself.
This definition has the problem that we haven’t introduced how to define functions on ($\lambda$-)terms recursively. Will fix in future.

**Lemma cr.4.**  If $M \xrightarrow{\beta} M'$ and $R \xrightarrow{\beta} R'$, then $M[R/y] \xrightarrow{\beta} M'[R'/y]$.

**Proof.** By induction on the derivation of $M \xrightarrow{\beta} M'$.

1. The last step is (1): Exercise.

2. The last step is (2): Then $M$ is $\lambda x. N$ and $M'$ is $\lambda x. N'$, where $N \xrightarrow{\beta} N'$. We want to prove that $(\lambda x. N)[R/y] \xrightarrow{\beta} (\lambda x. N')[R'/y]$, i.e., $\lambda x. N[R/y] \xrightarrow{\beta} \lambda x. N'[R/y]$. This follows immediately by (2) and the induction hypothesis.

3. The last step is (3): Exercise.

4. The last step is (4): $M$ is $(\lambda x. N)Q$ and $M'$ is $N'[Q'/x]$. We want to prove that $((\lambda x. N)Q)[R/y] \xrightarrow{\beta} N'[Q'/x][R'/y]$, i.e., $(\lambda x. N[R/y])Q[R/y] \xrightarrow{\beta} N'[R'/y][Q'[R'/y]/x]$. This follows by (4) and the induction hypothesis. 

**Problem cr.2.** Complete the proof of **Lemma cr.4**.

**Lemma cr.5.**  If $M \xrightarrow{\beta} M'$ then $M' \xrightarrow{\beta} M^{*\beta}$.

**Proof.** By induction on the derivation of $M \xrightarrow{\beta} M'$.

1. The last rule is (1): Exercise.

2. The last rule is (2): $M$ is $\lambda x. N$ and $M'$ is $\lambda x. N'$ with $N \xrightarrow{\beta} N'$. We want to show that $\lambda x. N' \xrightarrow{\beta} (\lambda x. N)^{*\beta}$, i.e., $\lambda x. N' \xrightarrow{\beta} \lambda x. N^{*\beta}$ by eq. (2). It follows by (2) and the induction hypothesis.

3. The last rule is (3): $M$ is $PQ$ and $M'$ is $P'Q'$ for some $P$, $Q$, $P'$ and $Q'$, with $P \xrightarrow{\beta} P'$ and $Q \xrightarrow{\beta} Q'$. By induction hypothesis, we have $P' \xrightarrow{\beta} P^{*\beta}$ and $Q' \xrightarrow{\beta} Q^{*\beta}$.

   a) If $P$ is $\lambda x. N$ for some $x$ and $N$, then $P'$ must be $\lambda x. N'$ for some $N'$ with $N \xrightarrow{\beta} N'$. By induction hypothesis we have $N' \xrightarrow{\beta} N^{*\beta}$ and $Q' \xrightarrow{\beta} Q^{*\beta}$. Then $(\lambda x. N')Q' \xrightarrow{\beta} N^{*\beta}[Q^{*\beta}/x]$ by (4).

   b) If $P$ is not a $\lambda$-abstract, then $P'Q' \xrightarrow{\beta} P^{*\beta}Q^{*\beta}$ by (3), and the right-hand side is $PQ^{*\beta}$ by eq. (3).
4. The last rule is (4): $M$ is $(\lambda x. N)Q$ and $M'$ is $N'[Q'/x]$ for some $x$, $N$, $Q$, $N'$, and $Q'$, with $N \xrightarrow{\beta} N'$ and $Q \xrightarrow{\beta} Q'$. By induction hypothesis we know $N' \xrightarrow{\beta} N^*\beta$ and $Q' \xrightarrow{\beta} Q^*\beta$. By Lemma cr.4 we have $N'[Q'/x] \xrightarrow{\beta} N^*\beta[Q^*\beta/x]$, the right-hand side of which is exactly $((\lambda x. N)Q)^*\beta$. \qed

**Problem cr.3.** Complete the proof of Lemma cr.5.

**Theorem cr.6.** $\xrightarrow{\beta}$ has the Church-Rosser property.

*Proof.* Immediate from Lemma cr.5. \qed

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Bibliography