


cr.1 Parallel $\beta\eta$-reduction

In this section we prove the Church-Rosser property for parallel $\beta\eta$-reduction, the parallel reduction notion corresponding to $\beta\eta$-reduction.

Definition cr.1 (Parallel $\beta\eta$-reduction, $\overset{\beta\eta}{\Rightarrow}$).

Parallel $\beta\eta$-reduction ($\overset{\beta\eta}{\Rightarrow}$) on terms is inductively defined as follows:

1. $x \overset{\beta\eta}{\Rightarrow} x$.
2. If $N \overset{\beta}{\Rightarrow} N'$ then $\lambda x. N \overset{\beta\eta}{\Rightarrow} \lambda x. N'$.
3. If $P \overset{\beta\eta}{\Rightarrow} P'$ and $Q \overset{\beta\eta}{\Rightarrow} Q'$ then $PQ \overset{\beta\eta}{\Rightarrow} P'Q'$.
4. If $N \overset{\beta\eta}{\Rightarrow} N'$ and $Q \overset{\beta\eta}{\Rightarrow} Q'$ then $(\lambda x. N)Q \overset{\beta\eta}{\Rightarrow} N'[Q'/x]$.
5. If $N \overset{\beta\eta}{\Rightarrow} N'$ then $\lambda x. Nx \overset{\beta\eta}{\Rightarrow} N'$, provided $x \notin FV(N)$.

Theorem cr.2. $M \overset{\beta\eta}{\Rightarrow} M$.

Proof. Exercise. \qed

Problem cr.1. Prove Theorem cr.2.

Definition cr.3 ($\beta\eta$-complete development).

The $\beta\eta$-complete development $M^*_{\beta\eta}$ of $M$ is defined as follows:

\[ x^{\beta\eta} = x \]  
\[ (\lambda x. N)^{\beta\eta} = \lambda x. N^{\beta\eta} \]  
\[ (PQ)^{\beta\eta} = P^{\beta\eta}Q^{\beta\eta} \]  
\[ ((\lambda x. N)Q)^{\beta\eta} = N^{\beta\eta}(Q^{\beta\eta}/x) \]  
\[ (\lambda x. Nx)^{\beta\eta} = N^{\beta\eta} \]  
\[ \text{if } x \notin FV(N) \]

Lemma cr.4. If $M \overset{\beta\eta}{\Rightarrow} M'$ and $R \overset{\beta\eta}{\Rightarrow} R'$, then $M[R/y] \overset{\beta\eta}{\Rightarrow} M'[R'/y]$.

Proof. By induction on the derivation of $M \overset{\beta\eta}{\Rightarrow} M'$.

The first four cases are exactly like those in ??$. If the last rule is (5), then $M$ is $\lambda x. N x$, $M'$ is $N'$ for some $x$ and $N'$ where $x \notin FV(N)$, and $N \overset{\beta\eta}{\Rightarrow} N'$.

We want to show that $(\lambda x. N x)[R/y] \overset{\beta\eta}{\Rightarrow} N'[R'/y]$, i.e., $\lambda x. N[R/y]x \overset{\beta\eta}{\Rightarrow} N'[R'/y]$. It follows by Definition cr.1(5) and the induction hypothesis. \qed

Lemma cr.5. If $M \overset{\beta\eta}{\Rightarrow} M'$ then $M' \overset{\beta\eta}{\Rightarrow} M^{\beta\eta}$.
Proof. By induction on the derivation of $M \xrightarrow{\beta\eta} M'$.

The first four cases are like those in ???. If the last rule is (5), then $M$ is $\lambda x. N x$ and $M'$ is $N'$ for some $x$, $N$, $N'$ where $x \notin FV(N)$ and $N \xrightarrow{\beta\eta} N'$. We want to show that $N' \xrightarrow{\beta\eta} (\lambda x. N x)^*\beta\eta$, i.e., $N' \xrightarrow{\beta\eta} N^*\beta\eta$, which is immediate by induction hypothesis.

Theorem cr.6. $\xrightarrow{\beta\eta}$ has the Church-Rosser property.

Proof. Immediate from Lemma cr.5.