

cr.1 Parallel $\beta\eta$ -reduction

`lam:cr:pbe:` In this section we prove the Church-Rosser property for parallel $\beta\eta$ -reduction,
`sec` the parallel reduction notion corresponding to $\beta\eta$ -reduction.

`lam:cr:pbe:` **Definition cr.1 (Parallel $\beta\eta$ -reduction, $\xRightarrow{\beta\eta}$).** Parallel $\beta\eta$ -reduction ($\xRightarrow{\beta\eta}$)
`defn:beredpar` on terms is inductively defined as follows:

- `lam:cr:pbe:` 1. $x \xRightarrow{\beta\eta} x$.
`defn:beredpar1`
- `lam:cr:pbe:` 2. If $N \xrightarrow{\beta} N'$ then $\lambda x. N \xRightarrow{\beta\eta} \lambda x. N'$.
`defn:beredpar2`
- `lam:cr:pbe:` 3. If $P \xRightarrow{\beta\eta} P'$ and $Q \xRightarrow{\beta\eta} Q'$ then $PQ \xRightarrow{\beta\eta} P'Q'$.
`defn:beredpar3`
- `lam:cr:pbe:` 4. If $N \xRightarrow{\beta\eta} N'$ and $Q \xRightarrow{\beta\eta} Q'$ then $(\lambda x. N)Q \xRightarrow{\beta\eta} N'[Q'/x]$.
`defn:beredpar4`
- `lam:cr:pbe:` 5. If $N \xRightarrow{\beta\eta} N'$ then $\lambda x. Nx \xRightarrow{\beta\eta} N'$, provided $x \notin FV(N)$.
`defn:beredpar5`

`lam:cr:pbe:` **Theorem cr.2.** $M \xRightarrow{\beta\eta} M$.
`thm:refl`

Proof. Exercise. □

Problem cr.1. Prove [Theorem cr.2](#).

`lam:cr:pbe:` **Definition cr.3 ($\beta\eta$ -complete development).** The $\beta\eta$ -complete develop-
`defn:becd` ment $M^{*\beta\eta}$ of M is defined as follows:

$$\text{lam:cr:pbe:} \quad x^{*\beta\eta} = x \tag{1}$$

$$\text{defn:becd1} \quad (\lambda x. N)^{*\beta\eta} = \lambda x. N^{*\beta\eta} \tag{2}$$

$$\text{lam:cr:pbe:} \quad (PQ)^{*\beta\eta} = P^{*\beta\eta}Q^{*\beta\eta} \tag{3}$$

$$\text{defn:becd3} \quad ((\lambda x. N)Q)^{*\beta\eta} = N^{*\beta\eta}[Q^{*\beta\eta}/x] \tag{4}$$

$$\text{lam:cr:pbe:} \quad (\lambda x. Nx)^{*\beta\eta} = N^{*\beta\eta} \tag{5}$$

`defn:becd5`

`lam:cr:pbe:` **Lemma cr.4.** If $M \xRightarrow{\beta\eta} M'$ and $R \xRightarrow{\beta\eta} R'$, then $M[R/y] \xRightarrow{\beta\eta} M'[R'/y]$.
`lem:comp`

Proof. By induction on the derivation of $M \xRightarrow{\beta\eta} M'$.

The first four cases are exactly like those in ???. If the last rule is (5), then M is $\lambda x. Nx$, M' is N' for some x and N' where $x \notin FV(N)$, and $N \xRightarrow{\beta\eta} N'$. We want to show that $(\lambda x. Nx)[R/y] \xRightarrow{\beta\eta} N'[R'/y]$, i.e., $\lambda x. N[R/y]x \xRightarrow{\beta\eta} N'[R'/y]$. It follows by [Definition cr.1\(5\)](#) and the induction hypothesis. □

`lam:cr:pbe:` **Lemma cr.5.** If $M \xRightarrow{\beta\eta} M'$ then $M' \xRightarrow{\beta\eta} M^{*\beta\eta}$.
`lem:cont`

Proof. By induction on the derivation of $M \xrightarrow{\beta\eta} M'$.

The first four cases are like those in ???. If the last rule is (5), then M is $\lambda x. Nx$ and M' is N' for some x, N, N' where $x \notin FV(N)$ and $N \xrightarrow{\beta\eta} N'$. We want to show that $N' \xrightarrow{\beta\eta} (\lambda x. Nx)^{*}\beta\eta$, i.e., $N' \xrightarrow{\beta\eta} N^{*\beta\eta}$, which is immediate by induction hypothesis. \square

Theorem cr.6. $\xrightarrow{\beta\eta}$ has the Church-Rosser property.

*lam:cr:pbe:
thm:cr*

Proof. Immediate from **Lemma cr.5**. \square

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Bibliography