\begin{section}{Parallel \(\beta\eta\)-reduction}

In this section we prove the Church-Rosser property for parallel \(\beta\eta\)-reduction, the parallel reduction notion corresponding to \(\beta\eta\)-reduction.

\begin{definition}[Parallel \(\beta\eta\)-reduction, \(\xrightarrow{\beta\eta}\)]
Parallel \(\beta\eta\)-reduction \(\xrightarrow{\beta\eta}\) on terms is inductively defined as follows:
\begin{enumerate}
\item \(x \xrightarrow{\beta\eta} x\).
\item If \(N \xrightarrow{\beta} N'\) then \(\lambda x. N \xrightarrow{\beta\eta} \lambda x. N'\).
\item If \(P \xrightarrow{\beta\eta} P'\) and \(Q \xrightarrow{\beta\eta} Q'\) then \(PQ \xrightarrow{\beta\eta} P'Q'\).
\item If \(N \xrightarrow{\beta\eta} N'\) and \(Q \xrightarrow{\beta\eta} Q'\) then \((\lambda x. N)Q \xrightarrow{\beta\eta} N'(Q'/x)\).
\item If \(N \xrightarrow{\beta\eta} N'\) then \(\lambda x. N x \xrightarrow{\beta\eta} N'\), provided \(x \notin \text{FV}(N)\).
\end{enumerate}
\end{definition}

\begin{theorem}
\(M \xrightarrow{\beta\eta} M\).
\end{theorem}

\begin{proof}
Exercise.
\end{proof}

\begin{problem}
Prove \textbf{Theorem cr.2}.
\end{problem}

\begin{definition}[\(\beta\eta\)-complete development]
The \(\beta\eta\)-complete development \(M^*_{\beta\eta}\) of \(M\) is defined as follows:
\begin{enumerate}
\item \(x^*_{\beta\eta} = x\).
\item \((\lambda x. N)^*_{\beta\eta} = \lambda x. N^*_{\beta\eta}\) \hspace{1cm} (1)
\item \((PQ)^*_{\beta\eta} = P^*_{\beta\eta}Q^*_{\beta\eta}\) if \(P\) is not a \(\lambda\)-abstract \hspace{1cm} (2)
\item \((\lambda x. N)Q^*_{\beta\eta} = N^*_{\beta\eta}(Q^*_{\beta\eta}/x)\) \hspace{1cm} (3)
\item \((\lambda x. N x)^*_{\beta\eta} = N^*_{\beta\eta}\) if \(x \notin \text{FV}(N)\) \hspace{1cm} (4)
\end{enumerate}
\end{definition}

\begin{lemma}
If \(M \xrightarrow{\beta\eta} M'\) and \(R \xrightarrow{\beta\eta} R'\), then \(M[R/y] \xrightarrow{\beta\eta} M'[R'/y]\).
\end{lemma}

\begin{proof}
By induction on the derivation of \(M \xrightarrow{\beta\eta} M'\).

The first four cases are exactly like those in ??.

If the last rule is (5), then \(M\) is \(\lambda x. N x\), \(M'\) is \(N'\) for some \(x\) and \(N'\) where \(x \notin \text{FV}(N)\), and \(N \xrightarrow{\beta\eta} N'\).

We want to show that \((\lambda x. N x)[R/y] \xrightarrow{\beta\eta} N'[R'/y]\), i.e., \(\lambda x. N[R/y]x \xrightarrow{\beta\eta} N'[R'/y]\).

It follows by \textbf{Definition cr.1}(5) and the induction hypothesis.
\end{proof}

\begin{lemma}
If \(M \xrightarrow{\beta\eta} M'\) then \(M' \xrightarrow{\beta\eta} M^*_{\beta\eta}\).
\end{lemma}

\end{section}
Proof. By induction on the derivation of $M \xrightarrow{\beta\eta} M'$.

The first four cases are like those in ???. If the last rule is (5), then $M$ is $\lambda x. Nx$ and $M'$ is $N'$ for some $x$, $N$, $N'$ where $x \notin FV(N)$ and $N \xrightarrow{\beta\eta} N'$. We want to show that $N' \xrightarrow{\beta\eta} (\lambda x. Nx)^{\beta\eta}$, i.e., $N' \xrightarrow{\beta\eta} N^{\beta\eta}$, which is immediate by induction hypothesis.

**Theorem cr.6.** $\xrightarrow{\beta\eta}$ has the Church-Rosser property.

*Proof.* Immediate from Lemma cr.5.

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Bibliography