In this section we prove the Church-Rosser property for parallel $\beta\eta$-reduction, the parallel reduction notion corresponding to $\beta\eta$-reduction.

**Definition cr.1 (Parallel $\beta\eta$-reduction, $\stackrel{\beta\eta}{\longrightarrow}$.** Parallel $\beta\eta$-reduction ($\stackrel{\beta\eta}{\longrightarrow}$) on terms is inductively defined as follows:

1. $x \stackrel{\beta\eta}{\longrightarrow} x$.
2. If $N \beta \rightarrow N'$ then $\lambda x. N \stackrel{\beta\eta}{\longrightarrow} \lambda x. N'$.
3. If $P \stackrel{\beta\eta}{\longrightarrow} P'$ and $Q \stackrel{\beta\eta}{\longrightarrow} Q'$ then $PQ \stackrel{\beta\eta}{\longrightarrow} P'Q'$.
4. If $N \beta\eta \rightarrow N'$ and $Q \beta\eta \rightarrow Q'$ then $(\lambda x. N)Q \stackrel{\beta\eta}{\longrightarrow} N'[Q'/x]$.
5. If $N \beta\eta \rightarrow N'$ then $\lambda x. Nx \stackrel{\beta\eta}{\longrightarrow} N'$, provided $x \notin \text{FV}(N)$.

**Theorem cr.2.** $M \stackrel{\beta\eta}{\rightarrow} M$.

**Proof.** Exercise. 

**Problem cr.1.** Prove Theorem cr.2.

**Definition cr.3 ($\beta\eta$-complete development).** The $\beta\eta$-complete development $M^{\ast\beta\eta}$ of $M$ is defined as follows:

1. $x^{\ast\beta\eta} = x$.
2. $(\lambda x. N)^{\ast\beta\eta} = \lambda x. N^{\ast\beta\eta}$
3. $(PQ)^{\ast\beta\eta} = P^{\ast\beta\eta}Q^{\ast\beta\eta}$ if $P$ is not a $\lambda$-abstract
4. $(\lambda x. N)^{\ast\beta\eta} = N^{\ast\beta\eta}[Q^{\ast\beta\eta}/x]$ if $x \notin \text{FV}(N)$

**Lemma cr.4.** If $M \beta\eta \rightarrow M'$ and $R \beta\eta \rightarrow R'$, then $M[R/y] \beta\eta \rightarrow M'[R'/y]$.

**Proof.** By induction on the derivation of $M \beta\eta \rightarrow M'$.

The first four cases are exactly like those in ??. If the last rule is (5), then $M$ is $\lambda x. N x$, $M'$ is $N'$ for some $x$ and $N'$ where $x \notin \text{FV}(N)$, and $N \beta\eta \rightarrow N'$.

We want to show that $(\lambda x. N x)[R/y] \beta\eta \rightarrow N'[R'/y]$, i.e., $\lambda x. N[R/y]x \beta\eta \rightarrow N'[R'/y]$. It follows by **Definition cr.1(5)** and the induction hypothesis.

**Lemma cr.5.** If $M \beta\eta \rightarrow M'$ then $M' \beta\eta \rightarrow M'^{\ast\beta\eta}$.
Proof. By induction on the derivation of $M \xrightarrow{\beta\eta} M'$.

The first four cases are like those in ???. If the last rule is (5), then $M$ is $\lambda x. N x$ and $M'$ is $N'$ for some $x, N, N'$ where $x \notin FV(N)$ and $N \xrightarrow{\beta\eta} N'$. We want to show that $N' \xrightarrow{\beta\eta} (\lambda x. N x)^\ast_{\beta\eta}$, i.e., $N' \xrightarrow{\beta\eta} N^\ast_{\beta\eta}$, which is immediate by induction hypothesis.

Theorem cr.6. $\xrightarrow{\beta\eta}$ has the Church-Rosser property.

Proof. Immediate from Lemma cr.5.

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Bibliography