In this chapter we introduce the concept of Church-Rosser property and some common properties of this property.

Definition cr.1 (Church-Rosser property, CR). A relation \( \rightarrow \) on terms is said to satisfy the Church-Rosser property iff, whenever \( M \rightarrow P \) and \( M \rightarrow Q \), then there exists some \( N \) such that \( P \rightarrow N \) and \( Q \rightarrow N \).

We can view the lambda calculus as a model of computation in which terms in normal form are “values” and a reducibility relation on terms are the “calculation rules.” The Church-Rosser property states is that when there is more than one way to proceed with a calculation, there is still only a single value of the expression.

To take an example from elementary algebra, there’s more than one way to calculate \( 4 \times (1 + 2) + 3 \). It can either be reduced to \( 4 \times 3 + 3 \) (if we first reduce \( 1 + 2 \) to \( 3 \)) or to \( 4 \times 1 + 4 \times 2 + 3 \) (if we first reduce \( 4 \times (1 + 2) \) using distributivity). Both of these, however, can be further reduced to \( 12 + 3 \).

If we take \( \rightarrow \) to be \( \beta \)-reduction, we easily see that a consequence of the Church-Rosser property is that if a term has a normal form, then it is unique. For suppose \( M \) can be reduced to \( P \) and \( Q \), both of which are normal forms. By Church-Rosser property, there exists some \( N \) such that both \( P \) and \( Q \) reduce to it. Since by assumption \( P \) and \( Q \) are normal forms, the reduction of \( P \) and \( Q \) to \( N \) can only be the trivial reduction, i.e., \( P \), \( Q \), and \( N \) are identical. This justifies our speaking of the normal form of a term.

In viewing the lambda calculus as a model of computation, then, the normal form of a term can be thought of as the “final result” of the computation starting with that term. The above corollary means there’s only one, if any, final result of a computation, just like there is only one result of computing \( 4 \times (1 + 2) + 3 \), namely 15.

Theorem cr.2. If a relation \( \rightarrow \) satisfies the Church-Rosser property, and \( \rightarrow \) is the smallest transitive relation containing \( \rightarrow \), then \( \rightarrow \) satisfies the Church-Rosser property too.

Proof. Suppose

\[
\begin{align*}
M & \rightarrow P_1 \rightarrow \ldots \rightarrow P_m \text{ and} \\
M & \rightarrow Q_1 \rightarrow \ldots \rightarrow Q_n.
\end{align*}
\]

We will prove the theorem by constructing a grid \( N \) of terms of height \( m + 1 \) and width \( n + 1 \). We use \( N_{i,j} \) to denote the term in the \( i \)-th row and \( j \)-th column.
We construct $N$ in such a way that $N_{i,j} \xrightarrow{X} N_{i+1,j}$ and $N_{i,j} \xrightarrow{X} N_{i,j+1}$. It is defined as follows:

\[
\begin{align*}
N_{0,0} &= M \\
N_{i,0} &= P_i & \text{if } 1 \leq i \leq m \\
N_{0,j} &= Q_j & \text{if } 1 \leq j \leq n
\end{align*}
\]

and otherwise:

\[
N_{i,j} = R
\]

where $R$ is a term such that $N_{i-1,j} \xrightarrow{X} R$ and $N_{i,j-1} \xrightarrow{X} R$. By the Church-Rosser property of $\xrightarrow{X}$, such a term always exists.

Now we have $N_{m,0} \xrightarrow{X} \ldots \xrightarrow{X} N_{m,n}$ and $N_{0,n} \xrightarrow{X} \ldots \xrightarrow{X} N_{m,n}$. Note $N_{m,0}$ is $P$ and $N_{0,n}$ is $Q$. By definition of $\xrightarrow{X}$ the theorem follows. \qed

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**Bibliography**