In this chapter we introduce the concept of Church–Rosser property and some common properties of this property.

**Definition 1 (Church–Rosser property, CR).** A relation $\rightarrow$ on terms is said to satisfy the *Church–Rosser property* iff, whenever $M \rightarrow P$ and $M \rightarrow Q$, then there exists some $N$ such that $P \rightarrow N$ and $Q \rightarrow N$.

We can view the lambda calculus as a model of computation in which terms in normal form are “values” and a reducibility relation on terms are the “calculation rules.” The Church–Rosser property states that when there is more than one way to proceed with a calculation, there is still only a single value of the expression.

To take an example from elementary algebra, there's more than one way to calculate $4 \times (1 + 2) + 3$. It can either be reduced to $4 \times 3 + 3$ (if we first reduce $1 + 2$ to $3$) or to $4 \times 1 + 4 \times 2 + 3$ (if we first reduce $4 \times (1 + 2)$ using distributivity). Both of these, however, can be further reduced to $12 + 3$.

If we take $\rightarrow$ to be $\beta$-reduction, we easily see that a consequence of the Church–Rosser property is that if a term has a normal form, then it is unique. For suppose $M$ can be reduced to $P$ and $Q$, both of which are normal forms. By the Church–Rosser property, there exists some $N$ such that both $P$ and $Q$ reduce to it. Since by assumption $P$ and $Q$ are normal forms, the reduction of $P$ and $Q$ to $N$ can only be the trivial reduction, i.e., $P$, $Q$, and $N$ are identical. This justifies our speaking of the normal form of a term.

In viewing the lambda calculus as a model of computation, then, the normal form of a term can be thought of as the “final result” of the computation starting with that term. The above corollary means there’s only one, if any, final result of a computation, just like there is only one result of computing $4 \times (1 + 2) + 3$, namely $15$.

**Theorem 2.** If a relation $\rightarrow$ satisfies the Church–Rosser property, and $\rightarrow^*$ is the smallest transitive relation containing $\rightarrow$, then $\rightarrow^*$ satisfies the Church–Rosser property too.

**Proof.** Suppose

$M \rightarrow P_1 \rightarrow \ldots \rightarrow P_m$ and $M \rightarrow Q_1 \rightarrow \ldots \rightarrow Q_n$.

We will prove the theorem by constructing a grid $N$ of terms of height is $m + 1$ and width $n + 1$. We use $N_{i,j}$ to denote the term in the $i$-th row and $j$-th column.
We construct \( N \) in such a way that \( N_{i,j} \xrightarrow{X} N_{i+1,j} \) and \( N_{i,j} \xrightarrow{X} N_{i,j+1} \). It is defined as follows:

\[
\begin{align*}
N_{0,0} &= M \\
N_{i,0} &= P_i & \text{if } 1 \leq i \leq m \\
N_{0,j} &= Q_j & \text{if } 1 \leq j \leq n
\end{align*}
\]

and otherwise:

\[
N_{i,j} = R
\]

where \( R \) is a term such that \( N_{i-1,j} \xrightarrow{X} R \) and \( N_{i,j-1} \xrightarrow{X} R \). By the Church–Rosser property of \( \xrightarrow{X} \), such a term always exists.

Now we have \( N_{m,0} \xrightarrow{X} \cdots \xrightarrow{X} N_{m,n} \) and \( N_{0,n} \xrightarrow{X} \cdots \xrightarrow{X} N_{m,n} \). Note \( N_{m,0} \) is \( P \) and \( N_{0,n} \) is \( Q \). By definition of \( \xrightarrow{X} \) the theorem follows. \( \square \)

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Bibliography