

Chapter udf

The Church-Rosser Property

cr.1 Definition and Properties

lam:cr:dap:
sec In this chapter we introduce the concept of Church-Rosser property and some common properties of this property.

Definition cr.1 (Church-Rosser property, CR). A relation \xrightarrow{X} on terms is said to satisfy the *Church-Rosser property* iff, whenever $M \xrightarrow{X} P$ and $M \xrightarrow{X} Q$, then there exists some N such that $P \xrightarrow{X} N$ and $Q \xrightarrow{X} N$.

We can view the lambda calculus as a model of computation in which terms in normal form are “values” and a reducibility relation on terms are the “calculation rules.” The Church-Rosser property states is that when there is more than one way to proceed with a calculation, there is still only a single value of the expression.

To take an example from elementary algebra, there’s more than one way to calculate $4 \times (1 + 2) + 3$. It can either be reduced to $4 \times 3 + 3$ (if we first reduce $1 + 2$ to 3) or to $4 \times 1 + 4 \times 2 + 3$ (if we first reduce $4 \times (1 + 2)$ using distributivity). Both of these, however, can be further reduced to $12 + 3$.

If we take \xrightarrow{X} to be β -reduction, we easily see that a consequence of the Church-Rosser property is that if a term has a normal form, then it is unique. For suppose M can be reduced to P and Q , both of which are normal forms. By Church-Rosser property, there exists some N such that both P and Q reduce to it. Since by assumption P and Q are normal forms, the reduction of P and Q to N can only be the trivial reduction, i.e., P , Q , and N are identical. This justifies our speaking of *the* normal form of a term.

In viewing the lambda calculus as a model of computation, then, the normal form of a term can be thought of as the “final result” of the computation starting with that term. The above corollary means there’s only one, if any, final result of a computation, just like there is only one result of computing $4 \times (1 + 2) + 3$, namely 15.

Theorem cr.2. *If a relation \xrightarrow{X} satisfies the Church-Rosser property, and \xrightarrow{X} is the smallest transitive relation containing \xrightarrow{X} , then \xrightarrow{X} satisfies the Church-Rosser property too.* lam:cr:dap:
thm:str

Proof. Suppose

$$\begin{aligned} M &\xrightarrow{X} P_1 \xrightarrow{X} \dots \xrightarrow{X} P_m \text{ and} \\ M &\xrightarrow{X} Q_1 \xrightarrow{X} \dots \xrightarrow{X} Q_n. \end{aligned}$$

We will prove the theorem by constructing a grid N of terms of height is $m + 1$ and width $n + 1$. We use $N_{i,j}$ to denote the term in the i -th row and j -th column.

We construct N in such a way that $N_{i,j} \xrightarrow{X} N_{i+1,j}$ and $N_{i,j} \xrightarrow{X} N_{i,j+1}$. It is defined as follows:

$$\begin{aligned} N_{0,0} &= M \\ N_{i,0} &= P_i && \text{if } 1 \leq i \leq m \\ N_{0,j} &= Q_j && \text{if } 1 \leq j \leq n \end{aligned}$$

and otherwise:

$$N_{i,j} = R$$

where R is a term such that $N_{i-1,j} \xrightarrow{X} R$ and $N_{i,j-1} \xrightarrow{X} R$. By the Church-Rosser property of \xrightarrow{X} , such a term always exists.

Now we have $N_{m,0} \xrightarrow{X} \dots \xrightarrow{X} N_{m,n}$ and $N_{0,n} \xrightarrow{X} \dots \xrightarrow{X} N_{m,n}$. Note $N_{m,0}$ is P and $N_{0,n}$ is Q . By definition of \xrightarrow{X} the theorem follows. □

cr.2 Parallel β -reduction

We introduce the notion of *parallel β -reduction*, and prove the it has the Church-Rosser property. lam:cr:pb:
sec

Definition cr.3 (parallel β -reduction, $\xRightarrow{\beta}$). Parallel reduction ($\xRightarrow{\beta}$) of terms is inductively defined as follows: lam:cr:pb:
defn:bredpar

1. $x \xRightarrow{\beta} x$. lam:cr:pb:
defn:bredpar1
2. If $N \xrightarrow{\beta} N'$ then $\lambda x. N \xRightarrow{\beta} \lambda x. N'$. lam:cr:pb:
defn:bredpar2
3. If $P \xRightarrow{\beta} P'$ and $Q \xRightarrow{\beta} Q'$ then $PQ \xRightarrow{\beta} P'Q'$. lam:cr:pb:
defn:bredpar3
4. If $N \xRightarrow{\beta} N'$ and $Q \xRightarrow{\beta} Q'$ then $(\lambda x. N)Q \xRightarrow{\beta} N'[Q'/x]$. lam:cr:pb:
defn:bredpar4

Parallel β -reduction allows us to reduce any number of redices in a term in one step. It is different from β -reduction in the sense that we can only contract redices that occur in the original term, but not redices arising from parallel β -reduction. For example, the term $(\lambda f. fx)(\lambda y. y)$ can only be parallel β -reduced to itself or to $(\lambda y. y)x$, but not further to x , although it β -reduces to x , because this redex arises only after one step of parallel β -reduction. A second parallel β -reduction step yields x , though.

lam:cr:pb: **Theorem cr.4.** $M \xRightarrow{\beta} M$.
thm:refl

Proof. Exercise. □

Problem cr.1. Prove **Theorem cr.4**.

lam:cr:pb: **Definition cr.5** (β -complete development). The β -complete development $M^{*\beta}$
defn:bed of M is defined inductively as follows:

lam:cr:pb: $x^{*\beta} = x$ (cr.1)
defn:bed1

lam:cr:pb: $(\lambda x. N)^{*\beta} = \lambda x. N^{*\beta}$ (cr.2)
defn:bed2

lam:cr:pb: $(PQ)^{*\beta} = P^{*\beta}Q^{*\beta}$ if P is not a λ -abstract (cr.3)
defn:bed3

lam:cr:pb: $((\lambda x. N)Q)^{*\beta} = N^{*\beta}[Q^{*\beta}/x]$ (cr.4)
defn:bed4

The β -complete development of a term, as its name suggests, is a “complete parallel reduction.” While for parallel β -reduction we still can choose to not contract a redex, for complete development we have no choice but to contract all of them. Thus the complete development of $(\lambda f. fx)(\lambda y. y)$ is $(\lambda y. y)x$, not itself.

This definition has the problem that we haven’t introduced how to define functions on (λ) -terms recursively. Will fix in future.

lam:cr:pb: **Lemma cr.6.** If $M \xRightarrow{\beta} M'$ and $R \xRightarrow{\beta} R'$, then $M[R/y] \xRightarrow{\beta} M'[R'/y]$.
lem:comp

Proof. By induction on the derivation of $M \xRightarrow{\beta} M'$.

1. The last step is (1): Exercise.
2. The last step is (2): Then M is $\lambda x. N$ and M' is $\lambda x. N'$, where $N \xRightarrow{\beta} N'$. We want to prove that $(\lambda x. N)[R/y] \xRightarrow{\beta} (\lambda x. N')[R'/y]$, i.e., $\lambda x. N[R/y] \xRightarrow{\beta} \lambda x. N'[R'/y]$. This follows immediately by (2) and the induction hypothesis.
3. The last step is (3): Exercise.

4. The last step is (4): M is $(\lambda x. N)Q$ and M' is $N'[Q'/x]$. We want to prove that $((\lambda x. N)Q)[R/y] \xRightarrow{\beta} N'[Q'/x][R'/y]$, i.e., $(\lambda x. N[R/y])Q[R/y] \xRightarrow{\beta} N'[R'/y][Q'[R'/y]/x]$. This follows by (4) and the induction hypothesis.

□

Problem cr.2. Complete the proof of [Lemma cr.6](#).

Lemma cr.7. *If $M \xRightarrow{\beta} M'$ then $M' \xRightarrow{\beta} M^{*\beta}$.*

[lam:cr:pb:](#)
[lem:cont](#)

Proof. By induction on the derivation of $M \xRightarrow{\beta} M'$.

1. The last rule is (1): Exercise.
2. The last rule is (2): M is $\lambda x. N$ and M' is $\lambda x. N'$ with $N \xRightarrow{\beta} N'$. We want to show that $\lambda x. N' \xRightarrow{\beta} (\lambda x. N)^{*\beta}$, i.e., $\lambda x. N' \xRightarrow{\beta} \lambda x. N^{*\beta}$ by [eq. \(cr.2\)](#). It follows by (2) and the induction hypothesis.
3. The last rule is (3): M is PQ and M' is $P'Q'$ for some P, Q, P' and Q' , with $P \xRightarrow{\beta} P'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis, we have $P' \xRightarrow{\beta} P^{*\beta}$ and $Q' \xRightarrow{\beta} Q^{*\beta}$.
 - a) If P is $\lambda x. N$ for some x and N , then P' must be $\lambda x. N'$ for some N' with $N \xRightarrow{\beta} N'$. By induction hypothesis we have $N' \xRightarrow{\beta} N^{*\beta}$ and $Q' \xRightarrow{\beta} Q^{*\beta}$. Then $(\lambda x. N')Q' \xRightarrow{\beta} N^{*\beta}[Q^{*\beta}/x]$ by (4).
 - b) If P is not a λ -abstract, then $P'Q' \xRightarrow{\beta} P^{*\beta}Q^{*\beta}$ by (3), and the right-hand side is $PQ^{*\beta}$ by [eq. \(cr.3\)](#).
4. The last rule is (4): M is $(\lambda x. N)Q$ and M' is $N'[Q'/x]$ for some x, N, Q, N' , and Q' , with $N \xRightarrow{\beta} N'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis we know $N' \xRightarrow{\beta} N^{*\beta}$ and $Q' \xRightarrow{\beta} Q^{*\beta}$. By [Lemma cr.6](#) we have $N'[Q'/x] \xRightarrow{\beta} N^{*\beta}[Q^{*\beta}/x]$, the right-hand side of which is exactly $((\lambda x. N)Q)^{*\beta}$.

□

Problem cr.3. Complete the proof of [Lemma cr.7](#).

Theorem cr.8. $\xRightarrow{\beta}$ has the Church-Rosser property.

[lam:cr:pb:](#)
[thm:cr](#)

Proof. Immediate from [Lemma cr.7](#).

□

cr.3 β -reduction

lam:cr:b:
sec
lam:cr:b:
lem:one-par

Lemma cr.9. *If $M \xrightarrow{\beta} M'$, then $M \xRightarrow{\beta} M'$.*

Proof. If $M \xrightarrow{\beta} M'$, then M is $(\lambda x. N)Q$, M' is $N[Q/x]$, for some x , N , and Q . Since $N \xRightarrow{\beta} N$ and $Q \xRightarrow{\beta} Q$ by **Theorem cr.4**, we immediately have $(\lambda x. N)Q \xRightarrow{\beta} N[Q/x]$ by **Definition cr.3(4)**. \square

lam:cr:b:
lem:par-red

Lemma cr.10. *If $M \xRightarrow{\beta} M'$, then $M \xrightarrow{\beta} M'$.*

Proof. By induction on the derivation of $M \xRightarrow{\beta} M'$.

1. The last rule is **(1)**: Then M and M' are just x , and $x \xrightarrow{\beta} x$.
2. The last rule is **(2)**: M is $\lambda x. N$ and M' is $\lambda x. N'$ for some x , N , N' , where $N \xRightarrow{\beta} N'$. By induction hypothesis we have $N \xrightarrow{\beta} N'$. Then $\lambda x. N \xrightarrow{\beta} \lambda x. N'$ (by the same series of $\xrightarrow{\beta}$ contractions as $N \xrightarrow{\beta} N'$).
3. The last rule is **(3)**: M is PQ and M' is $P'Q'$ for some P , Q , P' , Q' , where $P \xRightarrow{\beta} P'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis we have $P \xrightarrow{\beta} P'$ and $Q \xrightarrow{\beta} Q'$. So $PQ \xrightarrow{\beta} P'Q'$ by the reduction sequence $P \xrightarrow{\beta} P'$ followed by the reduction $Q \xrightarrow{\beta} Q'$.
4. The last rule is **(4)**: M is $(\lambda x. N)Q$ and M' is $N'[Q'/x]$ for some x , N , M' , Q , Q' , where $N \xRightarrow{\beta} N'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis we get $Q \xrightarrow{\beta} Q'$ and $N \xrightarrow{\beta} N'$. So $(\lambda x. N)Q \xrightarrow{\beta} N'[Q'/x]$ by $N \xrightarrow{\beta} N'$ followed by $Q \xrightarrow{\beta} Q'$ and finally contraction of $(\lambda x. N')Q'$ to $N'[Q'/x]$.

\square

lam:cr:b:
lem:str

Lemma cr.11. *$\xrightarrow{\beta}$ is the smallest transitive relation containing $\xRightarrow{\beta}$.*

Proof. Let \xrightarrow{X} be the smallest transitive relation containing $\xRightarrow{\beta}$.

$\xrightarrow{\beta} \subseteq \xrightarrow{X}$: Suppose $M \xrightarrow{\beta} M'$, i.e., $M \equiv M_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} M_k \equiv M'$. By **Lemma cr.9**, $M \equiv M_1 \xRightarrow{\beta} \dots \xRightarrow{\beta} M_k \equiv M'$. Since \xrightarrow{X} contains $\xRightarrow{\beta}$ and is transitive, $M \xrightarrow{X} M'$.

$\xrightarrow{X} \subseteq \xrightarrow{\beta}$: Suppose $M \xrightarrow{X} M'$, i.e., $M \equiv M_1 \xRightarrow{\beta} \dots \xRightarrow{\beta} M_k \equiv M'$. By **Lemma cr.10**, $M \equiv M_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} M_k \equiv M'$. Since $\xrightarrow{\beta}$ is transitive, $M \xrightarrow{\beta} M'$. \square

lam:cr:b:
thm:cr

Theorem cr.12. *$\xrightarrow{\beta}$ satisfies the Church-Rosser property.*

Proof. Immediate from **Theorem cr.2**, **Theorem cr.8**, and **Lemma cr.11**. \square

cr.4 Parallel $\beta\eta$ -reduction

In this section we prove the Church-Rosser property for parallel $\beta\eta$ -reduction, the parallel reduction notion corresponding to $\beta\eta$ -reduction. lam:cr:pbe:
sec

Definition cr.13 (Parallel $\beta\eta$ -reduction, $\xRightarrow{\beta\eta}$). *Parallel $\beta\eta$ -reduction* ($\xRightarrow{\beta\eta}$) on terms is inductively defined as follows: lam:cr:pbe:
defn:beredpar

1. $x \xRightarrow{\beta\eta} x$. lam:cr:pbe:
defn:beredpar1
2. If $N \xrightarrow{\beta} N'$ then $\lambda x. N \xRightarrow{\beta\eta} \lambda x. N'$. lam:cr:pbe:
defn:beredpar2
3. If $P \xRightarrow{\beta\eta} P'$ and $Q \xRightarrow{\beta\eta} Q'$ then $PQ \xRightarrow{\beta\eta} P'Q'$. lam:cr:pbe:
defn:beredpar3
4. If $N \xRightarrow{\beta\eta} N'$ and $Q \xRightarrow{\beta\eta} Q'$ then $(\lambda x. N)Q \xRightarrow{\beta\eta} N'[Q'/x]$. lam:cr:pbe:
defn:beredpar4
5. If $N \xRightarrow{\beta\eta} N'$ then $\lambda x. Nx \xRightarrow{\beta\eta} N'$, provided $x \notin FV(N)$. lam:cr:pbe:
defn:beredpar5

Theorem cr.14. $M \xRightarrow{\beta\eta} M$. lam:cr:pbe:
thm:refl

Proof. Exercise. □

Problem cr.4. Prove [Theorem cr.14](#).

Definition cr.15 ($\beta\eta$ -complete development). The $\beta\eta$ -complete development $M^{*\beta\eta}$ of M is defined as follows: lam:cr:pbe:
defn:becd

$$x^{*\beta\eta} = x \tag{cr.5} \quad \text{lam:cr:pbe: defn:becd1}$$

$$(\lambda x. N)^{*\beta\eta} = \lambda x. N^{*\beta\eta} \tag{cr.6} \quad \text{lam:cr:pbe: defn:becd2}$$

$$(PQ)^{*\beta\eta} = P^{*\beta\eta}Q^{*\beta\eta} \quad \text{if } P \text{ is not a } \lambda\text{-abstract} \tag{cr.7} \quad \text{lam:cr:pbe: defn:becd3}$$

$$((\lambda x. N)Q)^{*\beta\eta} = N^{*\beta\eta}[Q^{*\beta\eta}/x] \tag{cr.8} \quad \text{lam:cr:pbe: defn:becd4}$$

$$(\lambda x. Nx)^{*\beta\eta} = N^{*\beta\eta} \quad \text{if } x \notin FV(N) \tag{cr.9} \quad \text{lam:cr:pbe: defn:becd5}$$

Lemma cr.16. If $M \xRightarrow{\beta\eta} M'$ and $R \xRightarrow{\beta\eta} R'$, then $M[R/y] \xRightarrow{\beta\eta} M'[R'/y]$. lam:cr:pbe:
lem:comp

Proof. By induction on the derivation of $M \xRightarrow{\beta\eta} M'$.

The first four cases are exactly like those in [Lemma cr.6](#). If the last rule is (5), then M is $\lambda x. Nx$, M' is N' for some x and N' where $x \notin FV(N)$, and $N \xRightarrow{\beta\eta} N'$. We want to show that $(\lambda x. Nx)[R/y] \xRightarrow{\beta\eta} N'[R'/y]$, i.e., $\lambda x. N[R/y]x \xRightarrow{\beta\eta} N'[R'/y]$. It follows by [Definition cr.13\(5\)](#) and the induction hypothesis. □

Lemma cr.17. If $M \xRightarrow{\beta\eta} M'$ then $M' \xRightarrow{\beta\eta} M^{*\beta\eta}$. lam:cr:pbe:
lem:cont

Proof. By induction on the derivation of $M \xrightarrow{\beta\eta} M'$.

The first four cases are like those in [Lemma cr.7](#). If the last rule is (5), then M is $\lambda x.Nx$ and M' is N' for some x, N, N' where $x \notin FV(N)$ and $N \xrightarrow{\beta\eta} N'$. We want to show that $N' \xrightarrow{\beta\eta} (\lambda x.Nx)^{*}\beta\eta$, i.e., $N' \xrightarrow{\beta\eta} N^{*\beta\eta}$, which is immediate by induction hypothesis. \square

lam:cr:pb: thm:cr **Theorem cr.18.** $\xrightarrow{\beta\eta}$ has the Church-Rosser property.

Proof. Immediate from [Lemma cr.17](#). \square

cr.5 $\beta\eta$ -reduction

lam:cr:be: sec The Church-Rosser property holds for $\beta\eta$ -reduction ($\xrightarrow{\beta\eta}$).

lam:cr:be: lem:one-par **Lemma cr.19.** If $M \xrightarrow{\beta\eta} M'$, then $M \xrightarrow{\beta\eta} M'$.

Proof. By induction on the derivation of $M \xrightarrow{\beta\eta} M'$. If $M \xrightarrow{\beta} M'$ by η -conversion (i.e., ??), we use [Theorem cr.14](#). The other cases are as in [Lemma cr.9](#). \square

lam:cr:be: lem:par-red **Lemma cr.20.** If $M \xrightarrow{\beta\eta} M'$, then $M \xrightarrow{\beta\eta} M'$.

Proof. Induction on the derivation of $M \xrightarrow{\beta\eta} M'$.

If the last rule is (5), then M is $\lambda x.Nx$ and M' is N' for some x, N, N' where $x \notin FV(N)$ and $N \xrightarrow{\beta\eta} N'$. Thus we can first reduce $\lambda x.Nx$ to N by η -conversion, followed by the series of $\xrightarrow{\beta\eta}$ steps that show that $N \xrightarrow{\beta\eta} N'$, which holds by induction hypothesis. \square

lam:cr:be: lem:str **Lemma cr.21.** $\xrightarrow{\beta\eta}$ is the smallest transitive relation containing $\xrightarrow{\beta\eta}$.

Proof. As in [Lemma cr.11](#) \square

lam:cr:be: thm:cr **Theorem cr.22.** $\xrightarrow{\beta\eta}$ satisfies Church-Rosser property.

Proof. By [Theorem cr.2](#), [Theorem cr.18](#) and [Lemma cr.21](#). \square

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