Chapter udf

The Church-Rosser Property

cr.1 Definition and Properties

In this chapter we introduce the concept of Church-Rosser property and some common properties of this property.

Definition cr.1 (Church-Rosser property, CR). A relation \( \rightarrow \) on terms is said to satisfy the Church-Rosser property iff, whenever \( M \rightarrow P \) and \( M \rightarrow Q \), then there exists some \( N \) such that \( P \rightarrow N \) and \( Q \rightarrow N \).

We can view the lambda calculus as a model of computation in which terms in normal form are “values” and a reducibility relation on terms are the “calculation rules.” The Church-Rosser property states is that when there is more than one way to proceed with a calculation, there is still only a single value of the expression.

To take an example from elementary algebra, there’s more than one way to calculate \( 4 \times (1 + 2) + 3 \). It can either be reduced to \( 4 \times 3 + 3 \) (if we first reduce \( 1 + 2 \) to 3) or to \( 4 \times 1 + 4 \times 2 + 3 \) (if we first reduce \( 4 \times (1 + 2) \) using distributivity). Both of these, however, can be further reduced to \( 12 + 3 \).

If we take \( \rightarrow \) to be \( \beta \)-reduction, we easily see that a consequence of the Church-Rosser property is that if a term has a normal form, then it is unique. For suppose \( M \) can be reduced to \( P \) and \( Q \), both of which are normal forms. By Church-Rosser property, there exists some \( N \) such that both \( P \) and \( Q \) reduce to it. Since by assumption \( P \) and \( Q \) are normal forms, the reduction of \( P \) and \( Q \) to \( N \) can only be the trivial reduction, i.e., \( P \), \( Q \), and \( N \) are identical. This justifies our speaking of the normal form of a term.

In viewing the lambda calculus as a model of computation, then, the normal form of a term can be thought of as the “final result” of the computation starting with that term. The above corollary means there’s only one, if any, final result of a computation, just like there is only one result of computing \( 4 \times (1 + 2) + 3 \), namely 15.
Theorem cr.2. If a relation $\rightarrow_X$ satisfies the Church-Rosser property, and $\rightarrow \rightarrow_X$ is the smallest transitive relation containing $\rightarrow_X$, then $\rightarrow \rightarrow_X$ satisfies the Church-Rosser property too.

Proof. Suppose

$$M \rightarrow_X P_1 \rightarrow \ldots \rightarrow_X P_m \text{ and }$$
$$M \rightarrow_X Q_1 \rightarrow \ldots \rightarrow_X Q_n.$$

We will prove the theorem by constructing a grid $N$ of terms of height $m+1$ and width $n+1$. We use $N_{i,j}$ to denote the term in the $i$-th row and $j$-th column.

We construct $N$ in such a way that $N_{i,j} \rightarrow_X N_{i+1,j}$ and $N_{i,j} \rightarrow_X N_{i,j+1}$. It is defined as follows:

$$N_{0,0} = M$$
$$N_{i,0} = P_i \quad \text{if } 1 \leq i \leq m$$
$$N_{0,j} = Q_j \quad \text{if } 1 \leq j \leq n$$

and otherwise:

$$N_{i,j} = R$$

where $R$ is a term such that $N_{i-1,j} \rightarrow_X R$ and $N_{i,j-1} \rightarrow_X R$. By the Church-Rosser property of $\rightarrow_X$, such a term always exists.

Now we have $N_{m,0} \rightarrow_X \ldots \rightarrow_X N_{m,n}$ and $N_{0,n} \rightarrow_X \ldots \rightarrow_X N_{m,n}$. Note $N_{m,0}$ is $P$ and $N_{0,n}$ is $Q$. By definition of $\rightarrow \rightarrow_X$ the theorem follows. 

\hspace{1cm} $\square$

2 Parallel $\beta$-reduction

We introduce the notion of parallel $\beta$-reduction, and prove the it has the Church-Rosser property.

Definition cr.3 (parallel $\beta$-reduction, $\overset{\beta}{\Rightarrow}$). Parallel reduction ($\overset{\beta}{\Rightarrow}$) of terms is inductively defined as follows:

1. $x \overset{\beta}{\Rightarrow} x$.

2. If $N \overset{\beta}{\Rightarrow} N'$ then $\lambda x. N \overset{\beta}{\Rightarrow} \lambda x. N'$.

3. If $P \overset{\beta}{\Rightarrow} P'$ and $Q \overset{\beta}{\Rightarrow} Q'$ then $PQ \overset{\beta}{\Rightarrow} P'Q'$.

4. If $N \overset{\beta}{\Rightarrow} N'$ and $Q \overset{\beta}{\Rightarrow} Q'$ then $(\lambda x. N)Q \overset{\beta}{\Rightarrow} N'[Q'/x]$. 

\hspace{1cm} church-rosser rev. 666b46f (2020-02-13) by OLP / CC–BY
Parallel $\beta$-reduction allows us to reduce any number of redices in a term in one step. It is different from $\beta$-reduction in the sense that we can only contract redices that occur in the original term, but not redices arising from parallel $\beta$-reduction. For example, the term $(\lambda f. fx)(\lambda y.y)$ can only be parallel $\beta$-reduced to itself or to $(\lambda y.y)x$, but not further to $x$, although it $\beta$-reduces to $x$, because this redex arises only after one step of parallel $\beta$-reduction. A second parallel $\beta$-reduction step yields $x$, though.

**Theorem cr.4.** $M \xrightarrow{\beta} M$.

*Proof.* Exercise. $\square$

**Problem cr.1.** Prove Theorem cr.4.

**Definition cr.5 ($\beta$-complete development).** The $\beta$-complete development $M^\ast$ of $M$ is defined inductively as follows:

\begin{align*}
x^\ast &= x & \text{(cr.1)} \\
(\lambda x. N)^\ast &= \lambda x. N^\ast & \text{(cr.2)} \\
(PQ)^\ast &= P^\ast Q^\ast & \text{if } P \text{ is not a } \lambda\text{-abstract} & \text{(cr.3)} \\
(\lambda x. NQ)^\ast &= N^\ast [Q^\ast /x] & \text{(cr.4)}
\end{align*}

The $\beta$-complete development of a term, as its name suggests, is a “complete parallel reduction.” While for parallel $\beta$-reduction we still can choose to not contract a redex, for complete development we have no choice but to contract all of them. Thus the complete development of $(\lambda f. fx)(\lambda y.y)$ is $(\lambda y.y)x$, not itself.

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\begin{center}
This definition has the problem that we haven’t introduced how to define functions on (\lambda-)terms recursively. Will fix in future.
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**Lemma cr.6.** If $M \xrightarrow{\beta} M'$ and $R \xrightarrow{\beta} R'$, then $M[R/y] \xrightarrow{\beta} M'[R'/y]$.

*Proof.* By induction on the derivation of $M \xrightarrow{\beta} M'$.

1. The last step is (1): Exercise.

2. The last step is (2): Then $M$ is $\lambda x. N$ and $M'$ is $\lambda x. N'$, where $N \xrightarrow{\beta} N'$.

   We want to prove that $(\lambda x. N)[R/y] \xrightarrow{\beta} (\lambda x. N')[R'/y]$, i.e., $\lambda x. N[R/y] \xrightarrow{\beta} \lambda x. N'[R/y]$. This follows immediately by (2) and the induction hypothesis.

3. The last step is (3): Exercise.
4. The last step is (4): \( M \) is \((\lambda x. N)Q\) and \( M' \) is \( N'[Q'/x] \). We want to prove that \( ((\lambda x. N)Q)[R/y] \overset{\beta}{\rightarrow} N'[Q'/x][R'/y] \), i.e., \((\lambda x. N)[R/y])Q[R/y] \overset{\beta}{\rightarrow} N'[R'/y][Q'[R'/y]/x] \). This follows by (4) and the induction hypothesis. 

\[ \square \]

**Problem cr.2.** Complete the proof of Lemma cr.6.

**Lemma cr.7.** If \( M \overset{\beta}{\rightarrow} M' \) then \( M' \overset{\beta}{\rightarrow} M^{\ast\beta} \).

**Proof.** By induction on the derivation of \( M \overset{\beta}{\rightarrow} M' \).

1. The last rule is (1): Exercise.

2. The last rule is (2): \( M \) is \( \lambda x. N \) and \( M' \) is \( \lambda x. N' \) with \( N \overset{\beta}{\rightarrow} N' \). We want to show that \( \lambda x. N' \overset{\beta}{\rightarrow} (\lambda x. N)^{\ast\beta} \), i.e., \( \lambda x. N' \overset{\beta}{\rightarrow} \lambda x. N^{\ast\beta} \) by eq. (cr.2). It follows by (2) and the induction hypothesis.

3. The last rule is (3): \( M \) is \( PQ \) and \( M' \) is \( P'Q' \) for some \( P, Q, P' \) and \( Q' \), with \( P \overset{\beta}{\rightarrow} P' \) and \( Q \overset{\beta}{\rightarrow} Q' \). By induction hypothesis, we have \( P' \overset{\beta}{\rightarrow} P'^{\ast\beta} \) and \( Q' \overset{\beta}{\rightarrow} Q'^{\ast\beta} \).

   a) If \( P \) is \( \lambda x. N \) for some \( x \) and \( N \), then \( P' \) must be \( \lambda x. N' \) for some \( N' \) with \( N \overset{\beta}{\rightarrow} N' \). By induction hypothesis we have \( N' \overset{\beta}{\rightarrow} N'^{\ast\beta} \) and \( Q' \overset{\beta}{\rightarrow} Q'^{\ast\beta} \). Then \((\lambda x. N')Q' \overset{\beta}{\rightarrow} N'^{\ast\beta}[Q'^{\ast\beta}/x]\) by (4).

   b) If \( P \) is not a \( \lambda \)-abstract, then \( P'Q' \overset{\beta}{\rightarrow} P'^{\ast\beta}Q'^{\ast\beta} \) by (3), and the right-hand side is \( PQ'^{\ast\beta} \) by eq. (cr.3).

4. The last rule is (4): \( M \) is \((\lambda x. N)Q\) and \( M' \) is \( N'[Q'/x] \) for some \( x, N, Q, N', \) and \( Q' \), with \( N \overset{\beta}{\rightarrow} N' \) and \( Q \overset{\beta}{\rightarrow} Q' \). By induction hypothesis we know \( N' \overset{\beta}{\rightarrow} N'^{\ast\beta} \) and \( Q' \overset{\beta}{\rightarrow} Q'^{\ast\beta} \). By Lemma cr.6 we have \( N'[Q'/x] \overset{\beta}{\rightarrow} N'^{\ast\beta}[Q'^{\ast\beta}/x] \), the right-hand side of which is exactly \((\lambda x. N)Q)^{\ast\beta} \). 

\[ \square \]

**Problem cr.3.** Complete the proof of Lemma cr.7.

**Theorem cr.8.** \( \overset{\beta}{\rightarrow} \) has the Church-Rosser property.

**Proof.** Immediate from Lemma cr.7. 

\[ \square \]
**Lemma cr.9.** If \( M \xrightarrow{\beta} M' \), then \( M \xrightarrow{=} M' \).

*Proof.* If \( M \xrightarrow{\beta} M' \), then \( M \equiv (\lambda x.N)Q \), \( M' \equiv N[Q/x] \), for some \( x, N \), and \( Q \). Since \( N \xrightarrow{\beta} N \) and \( Q \xrightarrow{\beta} Q \) by Theorem cr.4, we immediately have \((\lambda x.N)Q \xrightarrow{\beta} N[Q/x]\) by Definition cr.3(4).

**Lemma cr.10.** If \( M \xrightarrow{\beta} M' \), then \( M \xrightarrow{\beta} M' \).

*Proof.* By induction on the derivation of \( M \xrightarrow{=} M' \).

1. The last rule is (1): Then \( M \) and \( M' \) are just \( x \) and \( x \xrightarrow{\beta} x \).
2. The last rule is (2): \( M \equiv \lambda x.N \) and \( M' \equiv \lambda x.N' \) for some \( x, N, N' \), where \( N \equiv \rightarrow N' \). By induction hypothesis we have \( N \xrightarrow{\beta} N' \). Then \( \lambda x.N \xrightarrow{\beta} \lambda x.N' \) (by the same series of \( \beta \) contractions as \( N \xrightarrow{\beta} N' \)).
3. The last rule is (3): \( M \equiv PQ \) and \( M' \equiv P'Q' \) for some \( P, Q, P', Q' \), where \( P \xrightarrow{\beta} P' \) and \( Q \xrightarrow{\beta} Q' \). By induction hypothesis we have \( P \xrightarrow{\beta} P' \) and \( Q \xrightarrow{\beta} Q' \). So \( PQ \xrightarrow{\beta} P'Q' \) by the reduction sequence \( P \xrightarrow{\beta} P' \) followed by the reduction \( Q \xrightarrow{\beta} Q' \).
4. The last rule is (4): \( M \equiv (\lambda x.N)Q \) and \( M' \equiv N'[Q'/x] \) for some \( x, N, M', Q, Q' \), where \( N \xrightarrow{\beta} N' \) and \( Q \xrightarrow{\beta} Q' \). By induction hypothesis we get \( Q \xrightarrow{\beta} Q' \) and \( N \xrightarrow{\beta} N' \). So \((\lambda x.N)Q \xrightarrow{\beta} N'[Q'/x]\) by \( N \xrightarrow{\beta} N' \) followed by \( Q \xrightarrow{\beta} Q' \) and finally contraction of \((\lambda x.N)Q' \) to \( N'[Q'/x] \).

**Lemma cr.11.** \( \xrightarrow{\beta} \) is the smallest transitive relation containing \( \xrightarrow{=} \).

*Proof.* Let \( \xrightarrow{X} \) be the smallest transitive relation containing \( \xrightarrow{=} \).

- \( \xrightarrow{\beta} \subseteq \xrightarrow{X} \): Suppose \( M \xrightarrow{\beta} M' \), i.e., \( M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M' \). By Lemma cr.9, \( M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M' \). Since \( \xrightarrow{X} \) contains \( \xrightarrow{=} \) and is transitive, \( M \xrightarrow{X} M' \).
- \( \xrightarrow{X} \subseteq \xrightarrow{\beta} \): Suppose \( M \xrightarrow{X} M' \), i.e., \( M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M' \). By Lemma cr.10, \( M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M' \). Since \( \xrightarrow{\beta} \) is transitive, \( M \xrightarrow{\beta} M' \).

**Theorem cr.12.** \( \xrightarrow{\beta} \) satisfies the Church-Rosser property.

*Proof.* Immediate from Theorem cr.2, Theorem cr.8, and Lemma cr.11.
cr.4 Parallel $\beta\eta$-reduction

In this section we prove the Church-Rosser property for parallel $\beta\eta$-reduction, the parallel reduction notion corresponding to $\beta\eta$-reduction.

**Definition cr.13 (Parallel $\beta\eta$-reduction, $\equiv_{\beta\eta}$).** Parallel $\beta\eta$-reduction ($\equiv_{\beta\eta}$) on terms is inductively defined as follows:

1. $x \equiv_{\beta\eta} x$.
2. If $N \equiv_{\beta} N'$ then $\lambda x. N \equiv_{\beta\eta} \lambda x. N'$.
3. If $P \equiv_{\beta\eta} P'$ and $Q \equiv_{\beta\eta} Q'$ then $PQ \equiv_{\beta\eta} P'Q'$.
4. If $N \equiv_{\beta\eta} N'$ and $Q \equiv_{\beta\eta} Q'$ then $(\lambda x. N)Q \equiv_{\beta\eta} N'[Q'/x]$.
5. If $N \equiv_{\beta\eta} N'$ then $\lambda x. Nx \equiv_{\beta\eta} N'$, provided $x \notin \text{FV}(N)$.

**Theorem cr.14.** $M \equiv_{\beta\eta} M$.

*Proof.* Exercise.

**Problem cr.4.** Prove Theorem cr.14.

**Definition cr.15 ($\beta\eta$-complete development).** The $\beta\eta$-complete development $M^{*}_{\beta\eta}$ of $M$ is defined as follows:

1. $x^{*}_{\beta\eta} = x$ (cr.5)
2. $(\lambda x. N)^{*}_{\beta\eta} = \lambda x. N^{*}_{\beta\eta}$ (cr.6)
3. $(PQ)^{*}_{\beta\eta} = P^{*}_{\beta\eta}Q^{*}_{\beta\eta}$ if $P$ is not a $\lambda$-abstract (cr.7)
4. $((\lambda x. N)Q)^{*}_{\beta\eta} = N^{*\beta\eta}[Q^{*\beta\eta}/x]$ (cr.8)
5. $(\lambda x. Nx)^{*}_{\beta\eta} = N^{*\beta\eta}$ if $x \notin \text{FV}(N)$ (cr.9)

**Lemma cr.16.** If $M \equiv_{\beta\eta} M'$ and $R \equiv_{\beta\eta} R'$, then $M[R/y] \equiv_{\beta\eta} M'[R'/y]$.

*Proof.* By induction on the derivation of $M \equiv_{\beta\eta} M'$.

The first four cases are exactly like those in Lemma cr.6. If the last rule is (5), then $M$ is $\lambda x. Nx$, $M'$ is $N'$ for some $x$ and $N'$ where $x \notin \text{FV}(N)$, and $N \equiv_{\beta\eta} N'$. We want to show that $(\lambda x. N)[R/y] \equiv_{\beta\eta} N'[R'/y]$, i.e., $\lambda x. N[R/y][x] \equiv_{\beta\eta} N'[R'/y]$. It follows by Definition cr.13(5) and the induction hypothesis.

**Lemma cr.17.** If $M \equiv_{\beta\eta} M'$ then $M' \equiv_{\beta\eta} M^{*\beta\eta}$.
Proof. By induction on the derivation of $M \beta\eta \Rightarrow M'$.

The first four cases are like those in Lemma cr.7. If the last rule is (5), then $M$ is $\lambda x. N x$ and $M'$ is $N'$ for some $x, N, N'$ where $x \notin FV(N)$ and $N \beta\eta N'$. We want to show that $N' \beta\eta (\lambda x. N x)^* \beta\eta$, i.e., $N' \beta\eta N^* \beta\eta$, which is immediate by induction hypothesis.

\textbf{Theorem cr.18.} $\beta\eta$ has the Church-Rosser property.

\textit{Proof.} Immediate from Lemma cr.17. \qed

\section{\textbf{cr.5} $\beta\eta$-reduction}

The Church-Rosser property holds for $\beta\eta$-reduction ($\beta\eta$).

\textbf{Lemma cr.19.} If $M \beta\eta M'$, then $M \beta\eta M'$.

\textit{Proof.} By induction on the derivation of $M \beta\eta M'$. If $M \beta M'$ by $\eta$-conversion (i.e., ??), we use Theorem cr.14. The other cases are as in Lemma cr.9. \qed

\textbf{Lemma cr.20.} If $M \beta\eta M'$, then $M \beta\eta \beta\eta M'$.

\textit{Proof.} Induction on the derivation of $M \beta\eta M'$.

If the last rule is (5), then $M$ is $\lambda x. N x$ and $M'$ is $N'$ for some $x, N, N'$ where $x \notin FV(N)$ and $N \beta\eta N'$. Thus we can first reduce $\lambda x. N x$ to $N$ by $\eta$-conversion, followed by the series of $\beta\eta$ steps that show that $N \beta\eta N'$, which holds by induction hypothesis. \qed

\textbf{Lemma cr.21.} $\beta\eta$ is the smallest transitive relation containing $\beta\eta$.

\textit{Proof.} As in Lemma cr.11. \qed

\textbf{Theorem cr.22.} $\beta\eta$ satisfies Church-Rosser property.

\textit{Proof.} By Theorem cr.2, Theorem cr.18 and Lemma cr.21. \qed

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Bibliography