Chapter udf

The Church–Rosser Property

cr.1 Definition and Properties

lam:cr:dap:sec In this chapter we introduce the concept of Church–Rosser property and some common properties of this property.

Definition cr.1 (Church–Rosser property, CR). A relation $\xrightarrow{X}$ on terms is said to satisfy the *Church–Rosser property* iff, whenever $M \xrightarrow{X} P$ and $M \xrightarrow{X} Q$, then there exists some N such that $P \xrightarrow{X} N$ and $Q \xrightarrow{X} N$.

We can view the lambda calculus as a model of computation in which terms in normal form are "values" and a reducibility relation on terms are the "calculation rules." The Church–Rosser property states is that when there is more than one way to proceed with a calculation, there is still only a single value of the expression.

To take an example from elementary algebra, there's more than one way to calculate $4 \times (1 + 2) + 3$. It can either be reduced to $4 \times 3 + 3$ (if we first reduce $1 + 2$ to 3) or to $4 \times 1 + 4 \times 2 + 3$ (if we first reduce $4 \times (1 + 2)$ using distributivity). Both of these, however, can be further reduced to $12 + 3$.

If we take $\xrightarrow{X}$ to be β-reduction, we easily see that a consequence of the Church–Rosser property is that if a term has a normal form, then it is unique. For suppose M can be reduced to P and Q, both of which are normal forms. By the Church–Rosser property, there exists some N such that both P and Q reduce to it. Since by assumption P and Q are normal forms, the reduction of P and Q to N can only be the trivial reduction, i.e., P, Q, and N are identical. This justifies our speaking of *the* normal form of a term.

In viewing the lambda calculus as a model of computation, then, the normal form of a term can be thought of as the "final result" of the computation starting with that term. The above corollary means there's only one, if any, final result of a computation, just like there is only one result of computing $4 \times (1 + 2) + 3$, namely 15.

Theorem cr.2. *If a relation $\xrightarrow{X}$ satisfies the Church–Rosser property, and $\xrightarrow{X}\!\!\!\!\rightarrow$ is the smallest transitive relation containing $\xrightarrow{X}$, then $\xrightarrow{X}\!\!\!\!\rightarrow$ satisfies the Church–Rosser property too.* lam:cr:dap:thm:str

Proof. Suppose

$$\begin{aligned}
&M \xrightarrow{X} P_1 \xrightarrow{X} \dots \xrightarrow{X} P_m \text{ and}\\
&M \xrightarrow{X} Q_1 \xrightarrow{X} \dots \xrightarrow{X} Q_n.
\end{aligned}$$

We will prove the theorem by constructing a grid N of terms of height is $m+1$ and width $n+1$. We use $N_{i,j}$ to denote the term in the i-th row and j-th column.

We construct N in such a way that $N_{i,j} \xrightarrow{X} N_{i+1,j}$ and $N_{i,j} \xrightarrow{X} N_{i,j+1}$. It is defined as follows:

$$\begin{aligned}
N_{0,0} &= M \\
N_{i,0} &= P_i && \text{if } 1 \le i \le m \\
N_{0,j} &= Q_j && \text{if } 1 \le j \le n
\end{aligned}$$

and otherwise:

$$N_{i,j} = R$$

where R is a term such that $N_{i-1,j} \xrightarrow{X} R$ and $N_{i,j-1} \xrightarrow{X} R$. By the Church–Rosser property of $\xrightarrow{X}$, such a term always exists.

Now we have $N_{m,0} \xrightarrow{X} \dots \xrightarrow{X} N_{m,n}$ and $N_{0,n} \xrightarrow{X} \dots \xrightarrow{X} N_{m,n}$. Note $N_{m,0}$ is P and $N_{0,n}$ is Q. By definition of $\xrightarrow{X}\!\!\!\!\rightarrow$ the theorem follows. □

cr.2 Parallel β-reduction

We introduce the notion of *parallel β-reduction*, and prove the it has the Church–Rosser property. lam:cr:pb:sec

Definition cr.3 (parallel β-reduction, $\overset{\beta}{\Longrightarrow}$). Parallel reduction ($\overset{\beta}{\Longrightarrow}$) of terms is inductively defined as follows: lam:cr:pb:defn:bredpar

1. $x \overset{\beta}{\Longrightarrow} x$. lam:cr:pb:defn:bredpar1
2. If $N \xrightarrow{\beta} N'$ then $\lambda x.\, N \overset{\beta}{\Longrightarrow} \lambda x.\, N'$. lam:cr:pb:defn:bredpar2
3. If $P \overset{\beta}{\Longrightarrow} P'$ and $Q \overset{\beta}{\Longrightarrow} Q'$ then $PQ \overset{\beta}{\Longrightarrow} P'Q'$. lam:cr:pb:defn:bredpar3
4. If $N \overset{\beta}{\Longrightarrow} N'$ and $Q \overset{\beta}{\Longrightarrow} Q'$ then $(\lambda x.\, N)Q \overset{\beta}{\Longrightarrow} N'[Q'/x]$. lam:cr:pb:defn:bredpar4

Parallel β-reduction allows us to reduce any number of redices in a term in one step. It is different from β-reduction in the sense that we can only contract redices that occur in the original term, but not redices arising from parallel β-reduction. For example, the term $(\lambda f.\, fx)(\lambda y.\, y)$ can only be parallel β-reduced to itself or to $(\lambda y.\, y)x$, but not further to x, although it β-reduces to x, because this redex arises only after one step of parallel β-reduction. A second parallel β-reduction step yields x, though.

lam:cr:pb: thm:refl **Theorem cr.4.** $M \overset{\beta}{\Longrightarrow} M$.

Proof. Exercise. □

Problem cr.1. Prove Theorem cr.4.

lam:cr:pb: defn:bcd **Definition cr.5 (β-complete development).** The *β-complete development* $M^{*\beta}$ of M is defined inductively as follows:

lam:cr:pb: defn:bcd1
$$x^{*\beta} = x \tag{cr.1}$$
lam:cr:pb: defn:bcd2
$$(\lambda x.\, N)^{*\beta} = \lambda x.\, N^{*\beta} \tag{cr.2}$$
lam:cr:pb: defn:bcd3
$$(PQ)^{*\beta} = P^{*\beta} Q^{*\beta} \qquad \text{if } P \text{ is not a } \lambda\text{-abstract} \tag{cr.3}$$
lam:cr:pb: defn:bcd4
$$((\lambda x.\, N)Q)^{*\beta} = N^{*\beta}[Q^{*\beta}/x] \tag{cr.4}$$

The β-complete development of a term, as its name suggests, is a "complete parallel reduction." While for parallel β-reduction we still can choose to not contract a redex, for complete development we have no choice but to contract all of them. Thus the complete development of $(\lambda f.\, fx)(\lambda y.\, y)$ is $(\lambda y.\, y)x$, not itself.

This definition has the problem that we haven't introduced how to define functions on (λ-)terms recursively. Will fix in future.

lam:cr:pb: lem:comp **Lemma cr.6.** *If $M \overset{\beta}{\Longrightarrow} M'$ and $R \overset{\beta}{\Longrightarrow} R'$, then $M[R/y] \overset{\beta}{\Longrightarrow} M'[R'/y]$.*

Proof. By induction on the derivation of $M \overset{\beta}{\Longrightarrow} M'$.

1. The last step is (1): Exercise.

2. The last step is (2): Then M is $\lambda x.\, N$ and M' is $\lambda x.\, N'$, where $N \overset{\beta}{\Longrightarrow} N'$. We want to prove that $(\lambda x.\, N)[R/y] \overset{\beta}{\Longrightarrow} (\lambda x.\, N')[R'/y]$, i.e., $\lambda x.\, N[R/y] \overset{\beta}{\Longrightarrow} \lambda x.\, N'[R/y]$. This follows immediately by (2) and the induction hypothesis.

3. The last step is (3): Exercise.

4. The last step is (4): M is $(\lambda x. N)Q$ and M' is $N'[Q'/x]$. We want to prove that $((\lambda x. N)Q)[R/y] \overset{\beta}{\Longrightarrow} N'[Q'/x][R'/y]$, i.e., $(\lambda x. N[R/y])Q[R/y] \overset{\beta}{\Longrightarrow} N'[R'/y][Q'[R'/y]/x]$. This follows by (4) and the induction hypothesis. □

Problem cr.2. Complete the proof of Lemma cr.6.

Lemma cr.7. *If $M \overset{\beta}{\Longrightarrow} M'$ then $M' \overset{\beta}{\Longrightarrow} M^{*\beta}$.* *lam:cr:pb:lem:cont*

Proof. By induction on the derivation of $M \overset{\beta}{\Longrightarrow} M'$.

1. The last rule is (1): Exercise.

2. The last rule is (2): M is $\lambda x. N$ and M' is $\lambda x. N'$ with $N \overset{\beta}{\Longrightarrow} N'$. We want to show that $\lambda x. N' \overset{\beta}{\Longrightarrow} (\lambda x. N)^{*\beta}$, i.e., $\lambda x. N' \overset{\beta}{\Longrightarrow} \lambda x. N^{*\beta}$ by eq. (cr.2). It follows by (2) and the induction hypothesis.

3. The last rule is (3):M is PQ and M' is $P'Q'$ for some P, Q, P' and Q', with $P \overset{\beta}{\Longrightarrow} P'$ and $Q \overset{\beta}{\Longrightarrow} Q'$. By induction hypothesis, we have $P' \overset{\beta}{\Longrightarrow} P^{*\beta}$ and $Q' \overset{\beta}{\Longrightarrow} Q^{*\beta}$.

   a) If P is $\lambda x. N$ for some x and N, then P' must be $\lambda x. N'$ for some N' with $N \overset{\beta}{\Longrightarrow} N'$. By induction hypothesis we have $N' \overset{\beta}{\Longrightarrow} N^{*\beta}$ and $Q' \overset{\beta}{\Longrightarrow} Q^{*\beta}$. Then $(\lambda x. N')Q' \overset{\beta}{\Longrightarrow} N^{*\beta}[Q^{*\beta}/x]$ by (4).

   b) If P is not a λ-abstract, then $P'Q' \overset{\beta}{\Longrightarrow} P^{*\beta}Q^{*\beta}$ by (3), and the right-hand side is $PQ^{*\beta}$ by eq. (cr.3).

4. The last rule is (4): M is $(\lambda x. N)Q$ and M' is $N'[Q'/x]$ for some x, N, Q, N', and Q', with $N \overset{\beta}{\Longrightarrow} N'$ and $Q \overset{\beta}{\Longrightarrow} Q'$. By induction hypothesis we know $N' \overset{\beta}{\Longrightarrow} N^{*\beta}$ and $Q' \overset{\beta}{\Longrightarrow} Q^{*\beta}$. By Lemma cr.6 we have $N'[Q'/x] \overset{\beta}{\Longrightarrow} N^{*\beta}[Q^{*\beta}/x]$, the right-hand side of which is exactly $((\lambda x. N)Q)^{*\beta}$. □

Problem cr.3. Complete the proof of Lemma cr.7.

Theorem cr.8. $\overset{\beta}{\Longrightarrow}$ *has the Church–Rosser property.* *lam:cr:pb:thm:cr*

Proof. Immediate from Lemma cr.7. □

cr.3 β-reduction

lam:cr:b: sec

lam:cr:b: lem:one-par **Lemma cr.9.** *If $M \xrightarrow{\beta} M'$, then $M \xRightarrow{\beta} M'$.*

Proof. If $M \xrightarrow{\beta} M'$, then M is $(\lambda x.\, N)Q$, M' is $N[Q/x]$, for some x, N, and Q. Since $N \xRightarrow{\beta} N$ and $Q \xRightarrow{\beta} Q$ by Theorem cr.4, we immediately have $(\lambda x.\, N)Q \xRightarrow{\beta} N[Q/x]$ by Definition cr.3(4). □

lam:cr:b: lem:par-red **Lemma cr.10.** *If $M \xRightarrow{\beta} M'$, then $M \xrightarrow{\beta}\!\!\!\!\rightarrow M'$.*

Proof. By induction on the derivation of $M \xRightarrow{\beta} M'$.

1. The last rule is (1): Then M and M' are just x, and $x \xrightarrow{\beta}\!\!\!\!\rightarrow x$.
2. The last rule is (2): M is $\lambda x.\, N$ and M' is $\lambda x.\, N'$ for some x, N, N', where $N \xRightarrow{\beta} N'$. By induction hypothesis we have $N \xrightarrow{\beta}\!\!\!\!\rightarrow N'$. Then $\lambda x.\, N \xrightarrow{\beta}\!\!\!\!\rightarrow \lambda x.\, N'$ (by the same series of $\xrightarrow{\beta}$ contractions as $N \xrightarrow{\beta}\!\!\!\!\rightarrow N'$).
3. The last rule is (3): M is PQ and M' is $P'Q'$ for some P, Q, P', Q', where $P \xRightarrow{\beta} P'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis we have $P \xrightarrow{\beta}\!\!\!\!\rightarrow P'$ and $Q \xrightarrow{\beta}\!\!\!\!\rightarrow Q'$. So $PQ \xrightarrow{\beta}\!\!\!\!\rightarrow P'Q'$ by the reduction sequence $P \xrightarrow{\beta}\!\!\!\!\rightarrow P'$ followed by the reduction $Q \xrightarrow{\beta}\!\!\!\!\rightarrow Q'$.
4. The last rule is (4): M is $(\lambda x.\, N)Q$ and M' is $N'[Q'/x]$ for some x, N, M', Q, Q', where $N \xRightarrow{\beta} N'$ and $Q \xRightarrow{\beta} Q'$. By induction hypothesis we get $Q \xrightarrow{\beta}\!\!\!\!\rightarrow Q'$ and $N \xrightarrow{\beta}\!\!\!\!\rightarrow N'$. So $(\lambda x.\, N)Q \xrightarrow{\beta}\!\!\!\!\rightarrow N'[Q'/x]$ by $N \xrightarrow{\beta}\!\!\!\!\rightarrow N'$ followed by $Q \xrightarrow{\beta}\!\!\!\!\rightarrow Q'$ and finally contraction of $(\lambda x.\, N')Q'$ to $N'[Q'/x]$. □

lam:cr:b: lem:str **Lemma cr.11.** *$\xrightarrow{\beta}\!\!\!\!\rightarrow$ is the smallest transitive relation containing $\xRightarrow{\beta}$.*

Proof. Let $\xrightarrow{X}\!\!\!\!\rightarrow$ be the smallest transitive relation containing $\xRightarrow{\beta}$.

$\xrightarrow{\beta}\!\!\!\!\rightarrow \subseteq \xrightarrow{X}\!\!\!\!\rightarrow$: Suppose $M \xrightarrow{\beta}\!\!\!\!\rightarrow M'$, i.e., $M \equiv M_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} M_k \equiv M'$. By Lemma cr.9, $M \equiv M_1 \xRightarrow{\beta} \dots \xRightarrow{\beta} M_k \equiv M'$. Since is $\xrightarrow{X}\!\!\!\!\rightarrow$ contains $\xRightarrow{\beta}$ and is transitive, $M \xrightarrow{X}\!\!\!\!\rightarrow M'$.

$\xrightarrow{X}\!\!\!\!\rightarrow \subseteq \xrightarrow{\beta}\!\!\!\!\rightarrow$: Suppose $M \xrightarrow{X}\!\!\!\!\rightarrow M'$, i.e., $M \equiv M_1 \xRightarrow{\beta} \dots \xRightarrow{\beta} M_k \equiv M'$. By Lemma cr.10, $M \equiv M_1 \xrightarrow{\beta}\!\!\!\!\rightarrow \dots \xrightarrow{\beta}\!\!\!\!\rightarrow M_k \equiv M'$. Since $\xrightarrow{\beta}\!\!\!\!\rightarrow$ is transitive, $M \xrightarrow{\beta}\!\!\!\!\rightarrow M'$. □

lam:cr:b: thm:cr **Theorem cr.12.** *$\xrightarrow{\beta}\!\!\!\!\rightarrow$ satisfies the Church–Rosser property.*

Proof. Immediate from Theorem cr.2, Theorem cr.8, and Lemma cr.11. □

cr.4 Parallel $\beta\eta$-reduction

In this section we prove the Church-Rosser property for parallel $\beta\eta$-reduction, the parallel reduction notion corresponding to $\beta\eta$-reduction. lam:cr:pbe:sec

Definition cr.13 (Parallel $\beta\eta$-reduction, $\xRightarrow{\beta\eta}$). *Parallel $\beta\eta$-reduction* ($\xRightarrow{\beta\eta}$) on terms is inductively defined as follows: lam:cr:pbe:defn:beredpar

1. $x \xRightarrow{\beta\eta} x$. lam:cr:pbe:defn:beredpar1
2. If $N \xrightarrow{\beta} N'$ then $\lambda x.\, N \xRightarrow{\beta\eta} \lambda x.\, N'$. lam:cr:pbe:defn:beredpar2
3. If $P \xRightarrow{\beta\eta} P'$ and $Q \xRightarrow{\beta\eta} Q'$ then $PQ \xRightarrow{\beta\eta} P'Q'$. lam:cr:pbe:defn:beredpar3
4. If $N \xRightarrow{\beta\eta} N'$ and $Q \xRightarrow{\beta\eta} Q'$ then $(\lambda x.\, N)Q \xRightarrow{\beta\eta} N'[Q'/x]$. lam:cr:pbe:defn:beredpar4
5. If $N \xRightarrow{\beta\eta} N'$ then $\lambda x.\, Nx \xRightarrow{\beta\eta} N'$, provided $x \notin FV(N)$. lam:cr:pbe:defn:beredpar5

Theorem cr.14. *$M \xRightarrow{\beta\eta} M$.* *lam:cr:pbe:thm:refl*

Proof. Exercise. □

Problem cr.4. Prove Theorem cr.14.

Definition cr.15 ($\beta\eta$-complete development). The *$\beta\eta$-complete development* $M^{*\beta\eta}$ of M is defined as follows: lam:cr:pbe:defn:becd

$$x^{*\beta\eta} = x \tag{cr.5}$$
lam:cr:pbe:defn:becd1

$$(\lambda x.\, N)^{*\beta\eta} = \lambda x.\, N^{*\beta\eta} \tag{cr.6}$$
lam:cr:pbe:defn:becd2

$$(PQ)^{*\beta\eta} = P^{*\beta\eta}Q^{*\beta\eta} \qquad \text{if } P \text{ is not a } \lambda\text{-abstract} \tag{cr.7}$$
lam:cr:pbe:defn:becd3

$$((\lambda x.\, N)Q)^{*\beta\eta} = N^{*\beta\eta}[Q^{*\beta\eta}/x] \tag{cr.8}$$
lam:cr:pbe:defn:becd4

$$(\lambda x.\, Nx)^{*\beta\eta} = N^{*\beta\eta} \qquad \text{if } x \notin FV(N) \tag{cr.9}$$
lam:cr:pbe:defn:becd5

Lemma cr.16. *If $M \xRightarrow{\beta\eta} M'$ and $R \xRightarrow{\beta\eta} R'$, then $M[R/y] \xRightarrow{\beta\eta} M'[R'/y]$.* *lam:cr:pbe:lem:comp*

Proof. By induction on the derivation of $M \xRightarrow{\beta\eta} M'$.

The first four cases are exactly like those in Lemma cr.6. If the last rule is (5), then M is $\lambda x.\, Nx$, M' is N' for some x and N' where $x \notin FV(N)$, and $N \xRightarrow{\beta\eta} N'$. We want to show that $(\lambda x.\, Nx)[R/y] \xRightarrow{\beta\eta} N'[R'/y]$, i.e., $\lambda x.\, N[R/y]x \xRightarrow{\beta\eta} N'[R'/y]$. It follows by Definition cr.13(5) and the induction hypothesis. □

Lemma cr.17. *If $M \xRightarrow{\beta\eta} M'$ then $M' \xRightarrow{\beta\eta} M^{*\beta\eta}$.* *lam:cr:pbe:lem:cont*

Proof. By induction on the derivation of $M \overset{\beta\eta}{\Longrightarrow} M'$.

The first four cases are like those in Lemma cr.7. If the last rule is (5), then M is $\lambda x.\, Nx$ and M' is N' for some x, N, N' where $x \notin FV(N)$ and $N \overset{\beta\eta}{\Longrightarrow} N'$. We want to show that $N' \overset{\beta\eta}{\Longrightarrow} (\lambda x.\, Nx)^{*\beta\eta}$, i.e., $N' \overset{\beta\eta}{\Longrightarrow} N^{*\beta\eta}$, which is immediate by induction hypothesis. □

lam:cr:pbe: thm:cr **Theorem cr.18.** *$\overset{\beta\eta}{\Longrightarrow}$ has the Church-Rosser property.*

Proof. Immediate from Lemma cr.17. □

cr.5 $\beta\eta$-reduction

lam:cr:be: sec The Church–Rosser property holds for $\beta\eta$-reduction ($\overset{\beta\eta}{\twoheadrightarrow}$).

lam:cr:be: lem:one-par **Lemma cr.19.** *If $M \xrightarrow{\beta\eta} M'$, then $M \overset{\beta\eta}{\Longrightarrow} M'$.*

Proof. By induction on the derivation of $M \xrightarrow{\beta\eta} M'$. If $M \xrightarrow{\beta} M'$ by η-conversion (i.e., **??**), we use Theorem cr.14. The other cases are as in Lemma cr.9. □

lam:cr:be: lem:par-red **Lemma cr.20.** *If $M \overset{\beta\eta}{\Longrightarrow} M'$, then $M \overset{\beta\eta}{\twoheadrightarrow} M'$.*

Proof. Induction on the derivation of $M \overset{\beta\eta}{\Longrightarrow} M'$.

If the last rule is (5), then M is $\lambda x.\, Nx$ and M' is N' for some x, N, N' where $x \notin FV(N)$ and $N \overset{\beta\eta}{\Longrightarrow} N'$. Thus we can first reduce $\lambda x.\, Nx$ to N by η-conversion, followed by the series of $\xrightarrow{\beta\eta}$ steps that show that $N \overset{\beta\eta}{\twoheadrightarrow} N'$, which holds by induction hypothesis. □

lam:cr:be: lem:str **Lemma cr.21.** *$\overset{\beta\eta}{\twoheadrightarrow}$ is the smallest transitive relation containing $\overset{\beta\eta}{\Longrightarrow}$.*

Proof. As in Lemma cr.11 □

lam:cr:be: thm:cr **Theorem cr.22.** *$\overset{\beta\eta}{\twoheadrightarrow}$ satisfies Church–Rosser property.*

Proof. By Theorem cr.2, Theorem cr.18 and Lemma cr.21. □

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