cr.1 $\beta$-reduction

**Lemma cr.1.** If \( M \rightarrow M' \), then \( M \Rightarrow M' \).

*Proof.* If \( M \rightarrow M' \), then \( M \) is \( (\lambda x. N)Q \), \( M' \) is \( N[Q/x] \), for some \( x \), \( N \), and \( Q \). Since \( N \Rightarrow N \) and \( Q \Rightarrow Q \) by ???, we immediately have \( (\lambda x. N)Q \Rightarrow N[Q/x] \) by ???.

**Lemma cr.2.** If \( M \rightarrow M' \), then \( M \rightarrow M' \).

*Proof.* By induction on the derivation of \( M \rightarrow M' \).

1. The last rule is ???: Then \( M \) and \( M' \) are just \( x \), and \( x \rightarrow x \).

2. The last rule is ???: \( M \) is \( \lambda x. N \) and \( M' \) is \( \lambda x. N' \) for some \( x \), \( N \), \( N' \), where \( N \Rightarrow N' \). By induction hypothesis we have \( N \rightarrow N' \). Then \( \lambda x. N \rightarrow \lambda x. N' \) (by the same series of \( \rightarrow \) contractions as \( N \rightarrow N' \)).

3. The last rule is ???: \( M \) is \( PQ \) and \( M' \) is \( P'Q' \) for some \( P \), \( Q \), \( P' \), \( Q' \), where \( P \rightarrow P' \) and \( Q \rightarrow Q' \). By induction hypothesis we have \( P \rightarrow P' \) and \( Q \rightarrow Q' \). So \( PQ \rightarrow P'Q' \) by the reduction sequence \( P \rightarrow P' \) followed by the reduction \( Q \rightarrow Q' \).

4. The last rule is ???: \( M \) is \( (\lambda x. N)Q \) and \( M' \) is \( N'[Q'/x] \) for some \( x \), \( N \), \( M' \), \( Q \), \( Q' \), where \( N \Rightarrow N' \) and \( Q \Rightarrow Q' \). By induction hypothesis we get \( Q \rightarrow Q' \) and \( N \rightarrow N' \). So \( (\lambda x. N)Q \rightarrow N'[Q'/x] \) by \( N \rightarrow N' \) followed by \( Q \rightarrow Q' \) and finally contraction of \( (\lambda x. N')Q' \) to \( N'[Q'/x] \).

**Lemma cr.3.** \( \rightarrow \) is the smallest transitive relation containing \( \Rightarrow \).

*Proof.* Let \( \bar{\rightarrow} \) be the smallest transitive relation containing \( \Rightarrow \).

\( \bar{\rightarrow} \subseteq \bar{\rightarrow} \): Suppose \( M \bar{\rightarrow} M' \), i.e., \( M \equiv M_1 \bar{\rightarrow} \ldots \bar{\rightarrow} M_k \equiv M' \). By **Lemma cr.1**, \( M \equiv M_1 \bar{\rightarrow} \ldots \bar{\rightarrow} M_k \equiv M' \). Since \( \bar{\rightarrow} \) contains \( \Rightarrow \) and is transitive, \( M \bar{\rightarrow} M' \).

\( \bar{\rightarrow} \subseteq \bar{\rightarrow} \): Suppose \( M \bar{\rightarrow} M' \), i.e., \( M \equiv M_1 \bar{\rightarrow} \ldots \bar{\rightarrow} M_k \equiv M' \). By **Lemma cr.2**, \( M \equiv M_1 \bar{\rightarrow} \ldots \bar{\rightarrow} M_k \equiv M' \). Since \( \bar{\rightarrow} \) is transitive, \( M \bar{\rightarrow} M' \).

**Theorem cr.4.** \( \bar{\rightarrow} \) satisfies the Church-Rosser property.
Proof. Immediate from ??, ??, and Lemma cr.3. □

Photo Credits

Bibliography