cr.1 β-reduction

lam:cr:b: sec

lam:cr:b: lem:one-par **Lemma cr.1.** *If $M \xrightarrow{\beta} M'$, then $M \overset{\beta}{\Longrightarrow} M'$.*

Proof. If $M \xrightarrow{\beta} M'$, then M is $(\lambda x.\, N)Q$, M' is $N[Q/x]$, for some x, N, and Q. Since $N \overset{\beta}{\Longrightarrow} N$ and $Q \overset{\beta}{\Longrightarrow} Q$ by **??**, we immediately have $(\lambda x.\, N)Q \overset{\beta}{\Longrightarrow} N[Q/x]$ by **????**. □

lam:cr:b: lem:par-red **Lemma cr.2.** *If $M \overset{\beta}{\Longrightarrow} M'$, then $M \overset{\beta}{\twoheadrightarrow} M'$.*

Proof. By induction on the derivation of $M \overset{\beta}{\Longrightarrow} M'$.

1. The last rule is **??**: Then M and M' are just x, and $x \overset{\beta}{\twoheadrightarrow} x$.
2. The last rule is **??**: M is $\lambda x.\, N$ and M' is $\lambda x.\, N'$ for some x, N, N', where $N \overset{\beta}{\Longrightarrow} N'$. By induction hypothesis we have $N \overset{\beta}{\twoheadrightarrow} N'$. Then $\lambda x.\, N \overset{\beta}{\twoheadrightarrow} \lambda x.\, N'$ (by the same series of $\xrightarrow{\beta}$ contractions as $N \overset{\beta}{\twoheadrightarrow} N'$).
3. The last rule is **??**: M is PQ and M' is $P'Q'$ for some P, Q, P', Q', where $P \overset{\beta}{\Longrightarrow} P'$ and $Q \overset{\beta}{\Longrightarrow} Q'$. By induction hypothesis we have $P \overset{\beta}{\twoheadrightarrow} P'$ and $Q \overset{\beta}{\twoheadrightarrow} Q'$. So $PQ \overset{\beta}{\twoheadrightarrow} P'Q'$ by the reduction sequence $P \overset{\beta}{\twoheadrightarrow} P'$ followed by the reduction $Q \overset{\beta}{\twoheadrightarrow} Q'$.
4. The last rule is **??**: M is $(\lambda x.\, N)Q$ and M' is $N'[Q'/x]$ for some x, N, M', Q, Q', where $N \overset{\beta}{\Longrightarrow} N'$ and $Q \overset{\beta}{\Longrightarrow} Q'$. By induction hypothesis we get $Q \overset{\beta}{\twoheadrightarrow} Q'$ and $N \overset{\beta}{\twoheadrightarrow} N'$. So $(\lambda x.\, N)Q \overset{\beta}{\twoheadrightarrow} N'[Q'/x]$ by $N \overset{\beta}{\twoheadrightarrow} N'$ followed by $Q \overset{\beta}{\twoheadrightarrow} Q'$ and finally contraction of $(\lambda x.\, N')Q'$ to $N'[Q'/x]$.

□

lam:cr:b: lem:str **Lemma cr.3.** *$\overset{\beta}{\twoheadrightarrow}$ is the smallest transitive relation containing $\overset{\beta}{\Longrightarrow}$.*

Proof. Let $\overset{X}{\twoheadrightarrow}$ be the smallest transitive relation containing $\overset{\beta}{\Longrightarrow}$.

$\overset{\beta}{\twoheadrightarrow} \subseteq \overset{X}{\twoheadrightarrow}$: Suppose $M \overset{\beta}{\twoheadrightarrow} M'$, i.e., $M \equiv M_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} M_k \equiv M'$. By Lemma cr.1, $M \equiv M_1 \overset{\beta}{\Longrightarrow} \dots \overset{\beta}{\Longrightarrow} M_k \equiv M'$. Since is $\overset{X}{\twoheadrightarrow}$ contains $\overset{\beta}{\Longrightarrow}$ and is transitive, $M \overset{X}{\twoheadrightarrow} M'$.

$\overset{X}{\twoheadrightarrow} \subseteq \overset{\beta}{\twoheadrightarrow}$: Suppose $M \overset{X}{\twoheadrightarrow} M'$, i.e., $M \equiv M_1 \overset{\beta}{\Longrightarrow} \dots \overset{\beta}{\Longrightarrow} M_k \equiv M'$. By Lemma cr.2, $M \equiv M_1 \overset{\beta}{\twoheadrightarrow} \dots \overset{\beta}{\twoheadrightarrow} M_k \equiv M'$. Since $\overset{\beta}{\twoheadrightarrow}$ is transitive, $M \overset{\beta}{\twoheadrightarrow} M'$. □

lam:cr:b: thm:cr **Theorem cr.4.** *$\overset{\beta}{\twoheadrightarrow}$ satisfies the Church-Rosser property.*

Proof. Immediate from **??**, **??**, and Lemma cr.3. □

Photo Credits

Bibliography