Lemma cr.1. If \( M \xrightarrow{\beta} M' \), then \( M \xrightarrow{=} M' \).

Proof. If \( M \xrightarrow{\beta} M' \), then \( M \) is \( (\lambda x. N)Q \), \( M' \) is \( N'[Q/x] \), for some \( x \), \( N \), and \( Q \). Since \( N \xrightarrow{=} N \) and \( Q \xrightarrow{=} Q \) by \( \beta \), we immediately have \( (\lambda x. N)Q \xrightarrow{=} N'[Q/x] \) by \( \beta \).

Lemma cr.2. If \( M \xrightarrow{\beta} M' \), then \( M \xrightarrow{\beta} M' \).

Proof. By induction on the derivation of \( M \xrightarrow{\beta} M' \).

1. The last rule is \( \beta \): Then \( M \) and \( M' \) are just \( x \), and \( x \xrightarrow{\beta} x \).

2. The last rule is \( \beta \): \( M \) is \( \lambda x. N \) and \( M' \) is \( \lambda x. N' \) for some \( x \), \( N \), \( N' \), where \( N \xrightarrow{=} N' \). By induction hypothesis we have \( N \xrightarrow{\beta} N' \). Then \( \lambda x. N \xrightarrow{\beta} \lambda x. N' \) (by the same series of \( \beta \) contractions as \( N \xrightarrow{\beta} N' \)).

3. The last rule is \( \beta \): \( M \) is \( PQ \) and \( M' \) is \( P'Q' \) for some \( P \), \( Q \), \( P' \), \( Q' \), where \( P \xrightarrow{\beta} P' \) and \( Q \xrightarrow{\beta} Q' \). By induction hypothesis we have \( P \xrightarrow{\beta} P' \) and \( Q \xrightarrow{\beta} Q' \). So \( PQ \xrightarrow{\beta} P'Q' \) by the reduction sequence \( P \xrightarrow{\beta} P' \) followed by the reduction \( Q \xrightarrow{\beta} Q' \).

4. The last rule is \( \beta \): \( M \) is \( (\lambda x. N)Q \) and \( M' \) is \( N'[Q'/x] \) for some \( x \), \( N \), \( M' \), \( Q \), \( Q' \), where \( N \xrightarrow{=} N' \) and \( Q \xrightarrow{\beta} Q' \). By induction hypothesis we get \( Q \xrightarrow{\beta} Q' \) and \( N \xrightarrow{\beta} N' \). So \( (\lambda x. N)Q \xrightarrow{\beta} N'[Q'/x] \) by \( N \xrightarrow{\beta} N' \) followed by \( Q \xrightarrow{\beta} Q' \) and finally contraction of \( (\lambda x. N)Q \) to \( N'[Q'/x] \).

Lemma cr.3. \( \xrightarrow{\beta} \) is the smallest transitive relation containing \( \xrightarrow{=} \).

Proof. Let \( X \xrightarrow{\beta} \) be the smallest transitive relation containing \( \xrightarrow{=} \).

\( \xrightarrow{\beta} \subseteq X \): Suppose \( M \xrightarrow{\beta} M' \), i.e., \( M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M' \). By Lemma cr.1, \( M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M' \). Since \( X \xrightarrow{\beta} \) contains \( \xrightarrow{=} \) and is transitive, \( M \xrightarrow{\beta} M' \).

\( X \subseteq \xrightarrow{\beta} \): Suppose \( M \xrightarrow{\beta} M' \), i.e., \( M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M' \). By Lemma cr.2, \( M \equiv M_1 \xrightarrow{\beta} \ldots \xrightarrow{\beta} M_k \equiv M' \). Since \( \xrightarrow{\beta} \) is transitive, \( M \xrightarrow{\beta} M' \).

Theorem cr.4. \( \xrightarrow{\beta} \) satisfies the Church–Rosser property.
Proof. Immediate from ??, ??, and Lemma cr.3. □

Photo Credits

Bibliography