

## cr.1 $\beta$ -reduction

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lem:one-par

**Lemma cr.1.** *If  $M \xrightarrow{\beta} M'$ , then  $M \xRightarrow{\beta} M'$ .*

*Proof.* If  $M \xrightarrow{\beta} M'$ , then  $M$  is  $(\lambda x. N)Q$ ,  $M'$  is  $N[Q/x]$ , for some  $x$ ,  $N$ , and  $Q$ . Since  $N \xRightarrow{\beta} N$  and  $Q \xRightarrow{\beta} Q$  by ??, we immediately have  $(\lambda x. N)Q \xRightarrow{\beta} N[Q/x]$  by ?????.  $\square$

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**Lemma cr.2.** *If  $M \xRightarrow{\beta} M'$ , then  $M \xrightarrow{\beta} M'$ .*

*Proof.* By induction on the derivation of  $M \xRightarrow{\beta} M'$ .

1. The last rule is ??: Then  $M$  and  $M'$  are just  $x$ , and  $x \xrightarrow{\beta} x$ .
2. The last rule is ??:  $M$  is  $\lambda x. N$  and  $M'$  is  $\lambda x. N'$  for some  $x$ ,  $N$ ,  $N'$ , where  $N \xRightarrow{\beta} N'$ . By induction hypothesis we have  $N \xrightarrow{\beta} N'$ . Then  $\lambda x. N \xrightarrow{\beta} \lambda x. N'$  (by the same series of  $\xrightarrow{\beta}$  contractions as  $N \xrightarrow{\beta} N'$ ).
3. The last rule is ??:  $M$  is  $PQ$  and  $M'$  is  $P'Q'$  for some  $P$ ,  $Q$ ,  $P'$ ,  $Q'$ , where  $P \xRightarrow{\beta} P'$  and  $Q \xRightarrow{\beta} Q'$ . By induction hypothesis we have  $P \xrightarrow{\beta} P'$  and  $Q \xrightarrow{\beta} Q'$ . So  $PQ \xrightarrow{\beta} P'Q'$  by the reduction sequence  $P \xrightarrow{\beta} P'$  followed by the reduction  $Q \xrightarrow{\beta} Q'$ .
4. The last rule is ??:  $M$  is  $(\lambda x. N)Q$  and  $M'$  is  $N'[Q'/x]$  for some  $x$ ,  $N$ ,  $M'$ ,  $Q$ ,  $Q'$ , where  $N \xRightarrow{\beta} N'$  and  $Q \xRightarrow{\beta} Q'$ . By induction hypothesis we get  $Q \xrightarrow{\beta} Q'$  and  $N \xrightarrow{\beta} N'$ . So  $(\lambda x. N)Q \xrightarrow{\beta} N'[Q'/x]$  by  $N \xrightarrow{\beta} N'$  followed by  $Q \xrightarrow{\beta} Q'$  and finally contraction of  $(\lambda x. N')Q'$  to  $N'[Q'/x]$ .  $\square$

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**Lemma cr.3.**  *$\xrightarrow{\beta}$  is the smallest transitive relation containing  $\xRightarrow{\beta}$ .*

*Proof.* Let  $\xrightarrow{X}$  be the smallest transitive relation containing  $\xRightarrow{\beta}$ .

$\xrightarrow{\beta} \subseteq \xrightarrow{X}$ : Suppose  $M \xrightarrow{\beta} M'$ , i.e.,  $M \equiv M_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} M_k \equiv M'$ . By **Lemma cr.1**,  $M \equiv M_1 \xRightarrow{\beta} \dots \xRightarrow{\beta} M_k \equiv M'$ . Since  $\xrightarrow{X}$  contains  $\xRightarrow{\beta}$  and is transitive,  $M \xrightarrow{X} M'$ .

$\xrightarrow{X} \subseteq \xrightarrow{\beta}$ : Suppose  $M \xrightarrow{X} M'$ , i.e.,  $M \equiv M_1 \xRightarrow{\beta} \dots \xRightarrow{\beta} M_k \equiv M'$ . By **Lemma cr.2**,  $M \equiv M_1 \xrightarrow{\beta} \dots \xrightarrow{\beta} M_k \equiv M'$ . Since  $\xrightarrow{\beta}$  is transitive,  $M \xrightarrow{\beta} M'$ .  $\square$

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**Theorem cr.4.**  *$\xrightarrow{\beta}$  satisfies the Church–Rosser property.*

*Proof.* Immediate from ??, ??, and **Lemma cr.3**.

□

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## Bibliography