The Church-Rosser property holds for $\beta\eta$-reduction ($\xrightarrow{\beta\eta}$).

**Lemma cr.1.** If $M \xrightarrow{\beta\eta} M'$, then $M \xrightarrow{\beta\eta} M'$.

*Proof.* By induction on the derivation of $M \xrightarrow{\beta\eta} M'$. If $M \xrightarrow{\beta} M'$ by $\eta$-conversion (i.e., $\Rightarrow$), we use $\Rightarrow$. The other cases are as in $\Rightarrow$. □

**Lemma cr.2.** If $M \xrightarrow{\beta\eta} M'$, then $M \xrightarrow{\beta\eta} M'$.

*Proof.* Induction on the derivation of $M \xrightarrow{\beta\eta} M'$.

If the last rule is $\Rightarrow$, then $M$ is $\lambda x. N x$ and $M'$ is $N'$ for some $x, N, N'$ where $x \notin FV(N)$ and $N \xrightarrow{\beta\eta} N'$. Thus we can first reduce $\lambda x. N x$ to $N$ by $\eta$-conversion, followed by the series of $\xrightarrow{\beta\eta}$ steps that show that $N \xrightarrow{\beta\eta} N'$, which holds by induction hypothesis. □

**Lemma cr.3.** $\xrightarrow{\beta\eta}$ is the smallest transitive relation containing $\xrightarrow{\beta\eta}$.

*Proof.* As in $\Rightarrow$. □

**Theorem cr.4.** $\xrightarrow{\beta\eta}$ satisfies Church-Rosser property.

*Proof.* By $\Rightarrow$, $\Rightarrow$ and Lemma cr.3. □

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Bibliography