# Chapter udf

# Intuitionistic Tableaux

Draft chapter on prefixed tableaux for intuitionistic logic. Needs more examples, completeness proofs, and discussion of how one can find countermodels from unsuccessful searches for closed tableaux.

## tab.1 Introduction

int:tab:int:sec Tableaux are certain (downward-branching) trees of signed formulas, i.e., pairs consisting of a truth value sign ($\mathbb{T}$ or $\mathbb{F}$) and a sentence

$$\mathbb{T}\,\varphi \text{ or } \mathbb{F}\,\varphi.$$

A tableau begins with a number of *assumptions*. Each further signed formula is generated by applying one of the inference rules. Some inference rules add one or more signed formulas to a tip of the tree; others add two new tips, resulting in two branches. Rules result in signed formulas where the formula is less complex than that of the signed formula to which it was applied. When a branch contains both $\mathbb{T}\,\varphi$ and $\mathbb{F}\,\varphi$, we say the branch is *closed*. If every branch in a tableau is closed, the entire tableau is closed. A closed tableau constitutes a derivation that shows that the set of signed formulas which were used to begin the tableau are unsatisfiable. This can be used to define a $\vdash$ relation: $\Gamma \vdash \varphi$ iff there is some finite set $\Gamma_0 = \{\psi_1, \dots, \psi_n\} \subseteq \Gamma$ such that there is a closed tableau for the assumptions

$$\{\mathbb{F}\,\varphi, \mathbb{T}\,\psi_1, \dots, \mathbb{T}\,\psi_n\}.$$

For intuitionistic logic, we have to both extend the notion of signed formula and adjust the rules for the connectives. In addition to a sign($\mathbb{T}$ or $\mathbb{F}$), formulas in modal tableaux also have *prefixes* $\sigma$. The prefixes are non-empty sequences of positive integers, i.e., $\sigma \in (\mathbb{Z}^+)^* \setminus \{\Lambda\}$. When we write such prefixes without the surrounding $\langle\ \rangle$, and separate the individual elements by .'s instead of ,'s.

| | |
|---|---|
| $\dfrac{\sigma\,\mathbb{T}\,\varphi \land \psi}{\begin{array}{c}\sigma\,\mathbb{T}\,\varphi \\ \sigma\,\mathbb{T}\,\psi\end{array}} \land\mathbb{T}$ | $\dfrac{\sigma\,\mathbb{F}\,\varphi \land \psi}{\sigma\,\mathbb{F}\,\varphi \quad \mid \quad \sigma\,\mathbb{F}\,\psi} \land\mathbb{F}$ |
| $\dfrac{\sigma\,\mathbb{T}\,\varphi \lor \psi}{\sigma\,\mathbb{T}\,\varphi \quad \mid \quad \sigma\,\mathbb{T}\,\psi} \lor\mathbb{T}$ | $\dfrac{\sigma\,\mathbb{F}\,\varphi \lor \psi}{\begin{array}{c}\sigma\,\mathbb{F}\,\varphi \\ \sigma\,\mathbb{F}\,\psi\end{array}} \lor\mathbb{F}$ |

Table tab.1: Prefixed tableau rules for $\land$ and $\lor$

int:tab:rul: tab:prop-rules

If $\sigma$ is a prefix, then $\sigma.n$ is $\sigma \frown \langle n \rangle$; e.g., if $\sigma = 1.2.1$, then $\sigma.3$ is 1.2.1.3. So for instance,

$$1.2\,\mathbb{T}\,\varphi \to (\psi \to \chi)$$

is a *prefixed signed formula* (or just a *prefixed formula* for short).

Intuitively, the prefix names a world in a model that might satisfy the formulas on a branch of a tableau, and if $\sigma$ names some world, then $\sigma.n$ names a world accessible from (the world named by) $\sigma$.

In intuitionistic models, the accessibility relation is reflexive and transitive. In terms of prefixes, this means that $\sigma$ is accessible from $\sigma$ itself, and so is any prefix that extends $\sigma$, i.e., any prefix of the form $\sigma.n_1.\cdots.n_k$. Let's introduce the notation $\sigma.*$ to indicate $\sigma$ itself and any extension of it. In other words, the prefixes $\sigma.*$ are all and only the prefixes accessible from $\sigma$.

## tab.2 Rules for Intuitionistic Logic

int:tab:rul: sec

The rules for the connectives $\land$ and $\lor$ are the same as for regular propositional signed tableaux, just with prefixes added. In each case, the rule applied to a signed formula $\sigma\,S\,\varphi$ produces new formulas that are also prefixed by $\sigma$. This should be intuitively clear: e.g., if $\varphi \land \psi$ is true at (a world named by) $\sigma$, then $\varphi$ and $\psi$ are true at $\sigma$ (and not at any other world). We collect the rules for $\land$ and $\lor$ in Table tab.1.

The closure condition is similar to that for ordinary tableaux, although we require that not just the formulas, but also that the prefixes must match. In fact, we can be somewhat more liberal: Since in intuitionistic models, formulas, once true, remain true, it is impossible that $\varphi$ is true at $\sigma$ but false at any accessible prefix $\sigma.*$. So a branch is closed if it contains both

$$\sigma\,\mathbb{T}\,\varphi \quad \text{and} \quad \sigma.*\,\mathbb{F}\,\varphi$$

for some prefix $\sigma$ and formula $\varphi$. Note that if the signs are reversed, i.e., if it contains

$$\sigma\,\mathbb{F}\,\varphi \quad \text{and} \quad \sigma.*\,\mathbb{T}\,\varphi$$

the branch is closed only if $*$ is the empty sequence.

| | |
|---|---|
| $\dfrac{\sigma\,\mathbb{T}\,\neg\varphi}{\sigma.{*}\,\mathbb{F}\,\varphi}\,\neg\mathbb{T}$ <br> $\sigma.{*}$ is used | $\dfrac{\sigma\,\mathbb{F}\,\neg\varphi}{\sigma.n\,\mathbb{T}\,\varphi}\,\neg\mathbb{F}$ <br> $\sigma.n$ is new |
| $\dfrac{\sigma\,\mathbb{T}\,\varphi\rightarrow\psi}{\sigma.{*}\,\mathbb{F}\,\varphi \quad \mid \quad \sigma.{*}\,\mathbb{T}\,\psi}\,{\rightarrow}\mathbb{T}$ <br> $\sigma.{*}$ is used | $\dfrac{\sigma\,\mathbb{F}\,\varphi\rightarrow\psi}{\begin{array}{c}\sigma.n\,\mathbb{T}\,\varphi\\ \sigma.n\,\mathbb{F}\,\psi\end{array}}\,{\rightarrow}\mathbb{F}$ <br> $\sigma.n$ is new |

Table tab.2: Prefixed tableau rules for $\neg$ and $\rightarrow$

int:tab:rul: tab:rules-lif-lnot

In addition, a branch is closed if it contains $\sigma\,\mathbb{T}\,\bot$.

The rules for setting up assumptions is also as for ordinary tableaux, except that for assumptions we always use the prefix 1. (It does not matter which prefix we use, as long as it's the same for all assumptions.) So, e.g., we say that

$$\psi_1, \dots, \psi_n \vdash \varphi$$

iff there is a closed tableau for the assumptions

$$1\,\mathbb{T}\,\psi_1, \dots, 1\,\mathbb{T}\,\psi_n, 1\,\mathbb{F}\,\varphi.$$

For the conditional $\rightarrow$, the rules differ from the classical and modal cases. The $\mathbb{T}{\rightarrow}$ rule extends a branch containing $\sigma\,\mathbb{T}\,\varphi\rightarrow\psi$ by $\sigma.{*}\,\mathbb{T}\,\varphi$ and $\sigma.{*}\,\mathbb{F}\,\psi$ on two different branches. It can only be applied for a prefix $\sigma.{*}$ which *already* occurs on the branch in which it is applied. Let's call such a prefix "used" (on the branch). (Since $\sigma.{*}$ includes $\sigma$ itself, the rule can always be applied by adding the prefixed signed formulas $\sigma\,\mathbb{T}\,\varphi$ and $\sigma\,\mathbb{F}\,\psi$ on separate branches.)

The $\mathbb{F}{\rightarrow}$ rule extends a branch containing $\sigma\,\mathbb{F}\,\varphi\rightarrow\psi$ by both $\sigma.n\,\mathbb{T}\,\varphi$ and $\sigma.n\,\mathbb{F}\,\psi$ on the same branch, with $\sigma.n$ a prefix new to the branch.

The rules for $\neg$ are defined analogously (using the definition of $\neg\varphi$ as $\varphi\rightarrow\bot$).

The rules are given in Table tab.2.

## tab.3 Tableaux for Intuitionistic Logic

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**Example tab.1.** We give a closed tableau that shows $(\varphi\wedge\psi)\rightarrow\chi \vdash \varphi\rightarrow(\psi\rightarrow\chi)$.

| | | | |
|---|---|---|---|
| 1. | $1\,\mathbb{T}\ (\varphi \land \psi) \to \chi$ | | Assumption |
| 2. | $1\,\mathbb{F}\ \varphi \to (\psi \to \chi)$ | | Assumption |
| 3. | $1.1\,\mathbb{T}\ \varphi$ | | $\to\mathbb{F}\,2$ |
| 4. | $1.1\,\mathbb{F}\ \psi \to \chi$ | | $\to\mathbb{F}\,2$ |
| 5. | $1.1.1\,\mathbb{T}\ \psi$ | | $\to\mathbb{F}\,4$ |
| 6. | $1.1.1\,\mathbb{F}\ \chi$ | | $\to\mathbb{F}\,4$ |
| 7. | $1.1.1\,\mathbb{F}\ \varphi \land \psi$ | $1.1.1\,\mathbb{T}\ \chi$ $\otimes$ | $\to\mathbb{T}\,1$ |
| 8. | $1.1.1\,\mathbb{F}\ \varphi$ $\otimes$ \| $1.1.1\,\mathbb{F}\ \psi$ $\otimes$ | | $\land\mathbb{F}\,4$ |

**Problem tab.1.** Find closed intuitionistic tableaux to show the following:

1. $\vdash \varphi \to (\psi \to \varphi)$
2. $\vdash \neg(\varphi \land \neg\varphi)$
3. $\varphi \to (\psi \to \chi) \vdash (\varphi \land \psi) \to \chi$
4. $\neg\varphi \lor \neg\psi \vdash \neg(\varphi \land \psi)$

## tab.4 Soundness for Intuitionistic Tableaux

explanation In order to show that intuitionistic tableaux are sound, we have to show that if int:tab:sou:sec

$$1\,\mathbb{T}\,\psi_1, \dots, 1\,\mathbb{T}\,\psi_n, 1\,\mathbb{F}\,\varphi$$

has a closed tableau then $\psi_1, \dots, \psi_n \vDash \varphi$. It is easier to prove the contrapositive: if for some $\mathfrak{M}$ and world $w$, $\mathfrak{M}, w \Vdash \psi_i$ for all $i = 1, \dots, n$ but $\mathfrak{M}, w \nVdash \varphi$, then no tableau can close. Such a countermodel shows that the initial assumptions of the tableau are satisfiable. The strategy of the proof is to show that whenever all the prefixed formulas on a tableau branch are satisfiable, any application of a rule results in at least one extended branch that is also satisfiable. Since closed branches are unsatisfiable, any tableau for a satisfiable set of prefixed formulas must have at least one open branch.

In order to apply this strategy in the modal case, we have to extend our definition of "satisfiable" to relational and prefixes. With that in hand, however, the proof is straightforward.

**Definition tab.2.** Let $P$ be some set of prefixes, i.e., $P \subseteq (\mathbb{Z}^+)^* \setminus \{\Lambda\}$ and let $\mathfrak{M}$ be a model. A function $f \colon P \to W$ is an *interpretation of* $P$ in $\mathfrak{M}$ if, whenever $\sigma$ and $\sigma.n$ are both in $P$, then $Rf(\sigma)f(\sigma.n)$.

Relative to an interpretation of prefixes $P$ we can define:

1. $\mathfrak{M}$ satisfies $\sigma\,\mathbb{T}\,\varphi$ iff $\mathfrak{M}, f(\sigma) \Vdash \varphi$.

2. $\mathfrak{M}$ satisfies $\sigma\,\mathbb{F}\,\varphi$ iff $\mathfrak{M}, f(\sigma) \nVdash \varphi$.

Note that since $R$ is reflexive and transitive and $\sigma.*$ denotes, $\sigma$, $\sigma.n_1$, $\sigma.n_1.n_2$, …, we also have that $Rf(\sigma)f(\sigma.*)$.

**Definition tab.3.** Let $\Gamma$ be a set of prefixed formulas, and let $P(\Gamma)$ be the set of prefixes that occur in it. If $f$ is an interpretation of $P(\Gamma)$ in $\mathfrak{M}$, we say that $\mathfrak{M}$ satisfies $\Gamma$ with respect to $f$, $\mathfrak{M}, f \Vdash \Gamma$, if $\mathfrak{M}$ satisfies every prefixed formula in $\Gamma$ with respect to $f$. $\Gamma$ is *satisfiable* iff there is a model $\mathfrak{M}$ and interpretation $f$ of $P(\Gamma)$ such that $\mathfrak{M}, f \Vdash \Gamma$.

**Proposition tab.4.** *If $\Gamma$ contains both $\sigma\,\mathbb{T}\,\varphi$ and $\sigma.*\,\mathbb{F}\,\varphi$ for some formula $\varphi$ and prefix $\sigma$, or it contains $\sigma\,\mathbb{T}\,\bot$, then $\Gamma$ is unsatisfiable.*

*Proof.* Since always $\mathfrak{M}, f(\sigma) \nVdash \bot$, a $\Gamma$ that contains $\mathbb{T}\,\bot$ is unsatisfiable.

There also cannot be a model $\mathfrak{M}$ and interpretation $f$ of $P(\Gamma)$ such that both If $\mathfrak{M}, f(\sigma) \Vdash \varphi$, then by **??**, since $Rf(\sigma)(\sigma.*)$, $\mathfrak{M}, f(\sigma) \Vdash \varphi$. So we cannot have both $\mathfrak{M}, f(\sigma) \Vdash \varphi$ and $\mathfrak{M}, f(\sigma.*) \nVdash \varphi$. □

*int:tab:sou: thm:tableau-soundness*

**Theorem tab.5 (Soundness).** *If $\Gamma$ has a closed tableau, $\Gamma$ is unsatisfiable.*

*Proof.* We call a branch of a tableau satisfiable iff the set of signed formulas on it is satisfiable, and let's call a tableau satisfiable if it contains at least one satisfiable branch.

We show the following: Extending a satisfiable tableau by one of the rules of inference always results in a satisfiable tableau. This will prove the theorem: any closed tableau results by applying rules of inference to the tableau consisting only of assumptions from $\Gamma$. So if $\Gamma$ were satisfiable, any tableau for it would be satisfiable. A closed tableau, however, is clearly not satisfiable, since all its branches are closed and closed branches are unsatisfiable.

Suppose we have a satisfiable tableau, i.e., a tableau with at least one satisfiable branch. Applying a rule of inference either adds signed formulas to a branch, or splits a branch in two. If the tableau has a satisfiable branch which is not extended by the rule application in question, it remains a satisfiable branch in the extended tableau, so the extended tableau is satisfiable. So we only have to consider the case where a rule is applied to a satisfiable branch.

Let $\Gamma$ be the set of signed formulas on that branch, and let $\sigma\,S\,\varphi \in \Gamma$ be the signed formula to which the rule is applied. If the rule does not result in a split branch, we have to show that the extended branch, i.e., $\Gamma$ together with the conclusions of the rule, is still satisfiable. If the rule results in split branch, we have to show that at least one of the two resulting branches is satisfiable.

1. The branch is expanded by applying $\neg\mathbb{T}$ to $\sigma\,\mathbb{T}\neg\psi \in \Gamma$. Then the extended branch contains the signed formulas $\Gamma \cup \{\sigma.*\,\mathbb{F}\,\psi\}$. Suppose $\mathfrak{M}, f \Vdash \Gamma$. In particular, $\mathfrak{M}, f(\sigma) \Vdash \neg\psi$. Thus, $\mathfrak{M}, w \nVdash \psi$ for any $w$ such that $Rf(\sigma)w$, and that includes $f(\sigma.*)$. So, $\mathfrak{M}$ satisfies $\sigma.*\,\mathbb{F}\,\psi$ with respect to $f$.

2. The branch is expanded by applying $\neg\mathbb{F}$ to $\sigma\,\mathbb{F}\,\neg\psi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\wedge\mathbb{T}$ to $\sigma\,\mathbb{T}\,\psi \wedge \chi \in \Gamma$, which results in two new signed formulas on the branch: $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \Vdash \psi \wedge \chi$. Then $\mathfrak{M}, f(\sigma) \Vdash \psi$ and $\mathfrak{M}, f(\sigma) \Vdash \chi$. This means that $\mathfrak{M}$ satisfies both $\sigma\,\mathbb{T}\,\psi$ and $\sigma\,\mathbb{T}\,\chi$ with respect to $f$.

4. The branch is expanded by applying $\vee\mathbb{F}$ to $\mathbb{F}\,\psi \vee \chi \in \Gamma$: Exercise.

5. The branch is expanded by applying $\rightarrow\mathbb{F}$ to $\sigma\,\mathbb{F}\,\psi \rightarrow \chi \in \Gamma$: This results in two new signed formulas on the branch: $\sigma.n\,\mathbb{T}\,\psi$ and $\sigma.n\,\mathbb{F}\,\chi$, where $\sigma.n$ is a new prefix on the branch, i.e., $\sigma.n \notin P(\Gamma)$.

   Since $\Gamma$ is satisfiable, there is a $\mathfrak{M}$ and interpretation $f$ of $P(\Gamma)$ such that $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \nVdash \psi \rightarrow \chi$. We have to show that $\Gamma \cup \{\sigma.n\,\mathbb{F}\,\psi \rightarrow \chi\}$ is satisfiable. To do this, we define an interpretation of $P(\Gamma) \cup \{\sigma.n\}$ as follows:

   Since $\mathfrak{M}, f(\sigma) \nVdash \psi \rightarrow \chi$, there is a $w \in W$ such that $Rf(\sigma)w$ such that $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \nVdash \chi$. Let $f'$ be like $f$, except that $f'(\sigma.n) = w$. Since $f'(\sigma) = f(\sigma)$ and $Rf(\sigma)w$, we have $Rf'(\sigma)f'(\sigma.n)$, so $f'$ is an interpretation of $P(\Gamma) \cup \{\sigma.n\}$. Obviously $\mathfrak{M}, f'(\sigma.n) \Vdash \psi$ and $\mathfrak{M}, f'(\sigma.n) \nVdash \chi$. Since $f(\sigma') = f'(\sigma')$ for all prefixes $\sigma' \in P(\Gamma)$, $\mathfrak{M}, f' \Vdash \Gamma$. So, $\mathfrak{M}, f'$ satisfies $\Gamma \cup \{\sigma.n\,\mathbb{F}\,\psi \rightarrow \chi\}$

Now let's consider the possible inferences with two premises.

1. The branch is expanded by applying $\wedge\mathbb{F}$ to $\sigma\,\mathbb{F}\,\psi \wedge \chi \in \Gamma$, which results in two branches, a left one continuing through $\sigma\,\mathbb{F}\,\psi$ and a right one through $\sigma\,\mathbb{F}\,\chi$. Suppose $\mathfrak{M}, f \Vdash \Gamma$, in particular $\mathfrak{M}, f(\sigma) \nVdash \psi \wedge \chi$. Then $\mathfrak{M}, f(\sigma) \nVdash \psi$ or $\mathfrak{M}, f(\sigma) \nVdash \chi$. In the former case, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\psi$, i.e., the left branch is satisfiable. In the latter, $\mathfrak{M}, f$ satisfies $\sigma\,\mathbb{F}\,\chi$, i.e., the right branch is satisfiable.

2. The branch is expanded by applying $\vee\mathbb{T}$ to $\sigma\,\mathbb{T}\,\psi \vee \chi \in \Gamma$: Exercise.

3. The branch is expanded by applying $\rightarrow\mathbb{T}$ to $\sigma\,\mathbb{T}\,\psi \rightarrow \chi \in \Gamma$: Exercise. □

**Problem tab.2.** Complete the proof of Theorem tab.5.

**Corollary tab.6.** *If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.* *int:tab:sou: cor:entailment-soundness*

*Proof.* If $\Gamma \vdash \varphi$ then for some $\psi_1, \dots, \psi_n \in \Gamma$, $\Delta = \{1\,\mathbb{F}\,\varphi, 1\,\mathbb{T}\,\psi_1, \dots, 1\,\mathbb{T}\,\psi_n\}$ has a closed tableau. We want to show that $\Gamma \vDash \varphi$. Suppose not, so for some $\mathfrak{M}$ and $w$, $\mathfrak{M}, w \Vdash \psi_i$ for $i = 1, \dots, n$, but $\mathfrak{M}, w \nVdash \varphi$. Let $f(1) = w$; then $f$ is an interpretation of $P(\Delta)$ into $\mathfrak{M}$, and $\mathfrak{M}$ satisfies $\Delta$ with respect to $f$. But by Theorem tab.5, $\Delta$ is unsatisfiable since it has a closed tableau, a contradiction. So we must have $\Gamma \vdash \varphi$ after all. □

int:tab:sou:
cor:weak-soundness

**Corollary tab.7.** *If* $\vdash \varphi$ *then* $\varphi$ *is true in all models.*

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# Bibliography