Lemma sc.1. If $\Delta$ is prime, then $\mathcal{M}(\Delta), \sigma \vDash \varphi$ iff $\Delta(\sigma) \vdash \varphi$.

Proof. By induction on $\varphi$.

1. $\varphi \equiv \bot$: Since $\Delta(\sigma)$ is prime, it is consistent, so $\Delta(\sigma) \not\vdash \varphi$. By definition, $\mathcal{M}(\Delta), \sigma \not\vDash \varphi$.

2. $\varphi \equiv p$: By definition of $\vDash$, $\mathcal{M}(\Delta), \sigma \vDash \varphi$ iff $\sigma \in V(p)$, i.e., $\Delta(\sigma) \vdash \varphi$.

3. $\varphi \equiv \neg \psi$: exercise.

4. $\varphi \equiv \psi \land \chi$: $\mathcal{M}(\Delta), \sigma \vDash \varphi$ iff $\mathcal{M}(\Delta), \sigma \vDash \psi$ and $\mathcal{M}(\Delta), \sigma \vDash \chi$. By induction hypothesis, $\mathcal{M}(\Delta), \sigma \vDash \psi$ if $\Delta(\sigma) \vdash \psi$, and similarly for $\chi$. But $\Delta(\sigma) \vdash \psi$ and $\Delta(\sigma) \vdash \chi$ iff $\Delta(\sigma) \vdash \varphi$.

5. $\varphi \equiv \psi \lor \chi$: First the contrapositive of the left-to-right direction: Assume $\Delta(\sigma) \not\vdash \psi \to \chi$. Then also $\Gamma \cup \{\psi\} \not\vdash \chi$. Since $(\psi, \chi)$ is $(\psi_n, \chi_n)$ for some $n$, we have $\Delta(\sigma.n) = (\Delta(\sigma) \cup \{\psi\})^*$, and $\Delta(\sigma.n) \not\vdash \psi$ but $\not\vdash \chi$. By inductive hypothesis, this holds iff $\Delta(\sigma) \not\vdash \psi$ of $\Delta(\sigma) \vdash \chi$. We have to show that this in turn holds iff $\Delta(\sigma) \vdash \varphi$. The left-to-right direction is clear. The right-to-left direction follows since $\Delta(\sigma)$ is prime.

6. $\varphi \equiv \psi \to \chi$: First the contrapositive of the left-to-right direction: Assume $\Delta(\sigma) \not\vdash \psi \to \chi$. Then also $\Gamma \cup (\psi \cup \{\psi\}) \not\vdash \chi$. Since $(\psi, \chi)$ is $(\psi_n, \chi_n)$ for some $n$, we have $\Delta(\sigma.n) = (\Delta(\sigma) \cup \{\psi\})^*$, and $\Delta(\sigma.n) \not\vdash \psi$ but $\not\vdash \chi$. By inductive hypothesis, this means that $\mathcal{M}(\Delta), \sigma \not\vDash \varphi$.

Now assume $\Delta(\sigma) \vdash \psi \to \chi$, and let $R\sigma$. Since $\Delta(\sigma) \not\subseteq \Delta(\sigma')$, we have: if $\Delta(\sigma') \vdash \psi$, then $\Delta(\sigma') \vdash \chi$. In other words, for every $\sigma'$ such that $R\sigma'$, either $\Delta(\sigma') \not\vdash \psi$ or $\Delta(\sigma') \vdash \chi$. By induction hypothesis, this means that whenever $R\sigma'$, either $\mathcal{M}(\Delta), \sigma' \not\vDash \psi$ or $\mathcal{M}(\Delta), \sigma' \vDash \chi$, i.e., $\mathcal{M}(\Delta), \sigma \vDash \varphi$. 

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Bibliography