

## sc.1 The Truth Lemma

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sec

int:sc:tru: **Lemma sc.1.** *If  $\Delta$  is prime, then  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$  iff  $\Delta(\sigma) \vdash \varphi$ .*

lem:truth

*Proof.* By induction on  $\varphi$ .

1.  $\varphi \equiv \perp$ : Since  $\Delta(\sigma)$  is prime, it is consistent, so  $\Delta(\sigma) \not\vdash \varphi$ . By definition,  $\mathfrak{M}(\Delta), \sigma \not\Vdash \varphi$ .
2.  $\varphi \equiv p$ : By definition of  $\Vdash$ ,  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$  iff  $\sigma \in V(p)$ , i.e.,  $\Delta(\sigma) \vdash \varphi$ .
3.  $\varphi \equiv \neg\psi$ : exercise.
4.  $\varphi \equiv \psi \wedge \chi$ :  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$  iff  $\mathfrak{M}(\Delta), \sigma \Vdash \psi$  and  $\mathfrak{M}(\Delta), \sigma \Vdash \chi$ . By induction hypothesis,  $\mathfrak{M}(\Delta), \sigma \Vdash \psi$  iff  $\Delta(\sigma) \vdash \psi$ , and similarly for  $\chi$ . But  $\Delta(\sigma) \vdash \psi$  and  $\Delta(\sigma) \vdash \chi$  iff  $\Delta(\sigma) \vdash \varphi$ .
5.  $\varphi \equiv \psi \vee \chi$ :  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$  iff  $\mathfrak{M}(\Delta), \sigma \Vdash \psi$  or  $\mathfrak{M}(\Delta), \sigma \Vdash \chi$ . By induction hypothesis, this holds iff  $\Delta(\sigma) \vdash \psi$  or  $\Delta(\sigma) \vdash \chi$ . We have to show that this in turn holds iff  $\Delta(\sigma) \vdash \varphi$ . The left-to-right direction is clear. The right-to-left direction follows since  $\Delta(\sigma)$  is prime.
6.  $\varphi \equiv \psi \rightarrow \chi$ : First the contrapositive of the left-to-right direction: Assume  $\Delta(\sigma) \not\vdash \psi \rightarrow \chi$ . Then also  $\Gamma * (\sigma) \cup \{\psi\} \not\vdash \chi$ . Since  $\langle \psi, \chi \rangle$  is  $\langle \psi_n, \chi_n \rangle$  for some  $n$ , we have  $\Delta(\sigma.n) = (\Delta(\sigma) \cup \{\psi\})^*$ , and  $\Delta(\sigma.n) \vdash \psi$  but  $\not\vdash \chi$ . By inductive hypothesis,  $\mathfrak{M}(\Delta), \sigma.n \Vdash \psi$  and  $\mathfrak{M}(\Delta), \sigma.n \not\Vdash \chi$ . Since  $R\sigma(\sigma.n)$ , this means that  $\mathfrak{M}(\Delta), \sigma \not\Vdash \varphi$ .  
Now assume  $\Delta(\sigma) \vdash \psi \rightarrow \chi$ , and let  $R\sigma\sigma'$ . Since  $\Delta(\sigma) \subseteq \Delta(\sigma')$ , we have: if  $\Delta(\sigma') \vdash \psi$ , then  $\Delta(\sigma') \vdash \chi$ . In other words, for every  $\sigma'$  such that  $R\sigma\sigma'$ , either  $\Delta(\sigma') \not\vdash \psi$  or  $\Delta(\sigma') \vdash \chi$ . By induction hypothesis, this means that whenever  $R\sigma\sigma'$ , either  $\mathfrak{M}(\Delta), \sigma' \not\Vdash \psi$  or  $\mathfrak{M}(\Delta), \sigma' \Vdash \chi$ , i.e.,  $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ .

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## Bibliography