sc.1 Soundness of Natural Deduction

int:sc:snd: sec

We will now prove soundness of natural deduction with regards to the relational semantics, that is, showing that if a formula is derivable from a set of assumptions then the set of assumptions entails the formula.

int:sc:snd: thm:soundness

Theorem sc.1 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*

Proof. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the derivation of φ from Γ.

1. If the derivation consists of just the assumption φ, we have $\varphi \vdash \varphi$, and want to show that $\varphi \vDash \varphi$. Suppose that $\mathfrak{M}, w \Vdash \varphi$. Then trivially $\mathfrak{M}, w \Vdash \varphi$.

2. The derivation ends in $\wedge$Intro: The derivations of the premises ψ from undischarged assumptions Γ and of χ from undischarged assumptions Δ show that $\Gamma \vdash \psi$ and $\Delta \vdash \chi$. By induction hypothesis we have that $\Gamma \vDash \psi$ and $\Delta \vDash \chi$. We have to show that $\Gamma \cup \Delta \vDash \varphi \wedge \psi$, since the undischarged assumptions of the entire derivation are Γ together with Δ. So suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$. Then also $\mathfrak{M}, w \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. Similarly, $\mathfrak{M}, w \Vdash \chi$. So $\mathfrak{M}, w \Vdash \psi \wedge \chi$.

3. The derivation ends in $\wedge$Elim: The derivation of the premise $\psi \wedge \chi$ from undischarged assumptions Γ shows that $\Gamma \vdash \psi \wedge \chi$. By induction hypothesis, $\Gamma \vDash \psi \wedge \chi$. We have to show that $\Gamma \vDash \psi$. So suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\Gamma \vDash \psi \wedge \chi$, $\mathfrak{M}, w \Vdash \psi \wedge \chi$. Then also $\mathfrak{M}, w \Vdash \psi$. Similarly if $\wedge$Elim ends in χ, then $\Gamma \vDash \chi$.

4. The derivation ends in $\vee$Intro: Suppose the premise is ψ, and the undischarged assumptions of the derivation ending in ψ are Γ. Then we have $\Gamma \vdash \psi$ and by inductive hypothesis, $\Gamma \vDash \psi$. We have to show that $\Gamma \vDash \psi \vee \chi$. Suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. But then also $\mathfrak{M}, w \Vdash \psi \vee \chi$. Similarly, if the premise is χ, we have that $\Gamma \vDash \chi$.

5. The derivation ends in $\vee$Elim: The derivations ending in the premises are of $\psi \vee \chi$ from undischarged assumptions Γ, of θ from undischarged assumptions $\Delta_1 \cup \{\psi\}$, and of θ from undischarged assumptions $\Delta_2 \cup \{\chi\}$. So we have $\Gamma \vdash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vdash \theta$, and $\Delta_2 \cup \{\chi\} \vdash \theta$. By induction hypothesis, $\Gamma \vDash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vDash \theta$, and $\Delta_2 \cup \{\chi\} \vDash \theta$. We have to prove that $\Gamma \cup \Delta_1 \cup \Delta_2 \vDash \theta$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta_1 \cup \Delta_2$. Then $\mathfrak{M}, w \Vdash \Gamma$ and since $\Gamma \vDash \psi \vee \chi$, $\mathfrak{M}, w \Vdash \psi \vee \chi$. By definition of $\mathfrak{M} \Vdash$, either $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$. So we distinguish cases: (a) $\mathfrak{M} \Vdash \psi[\mathrm{w}]$. Then $\mathfrak{M}, w \Vdash \Delta_1 \cup \{\psi\}$. Since $\Delta_1 \cup \psi \vDash \theta$, we have $\mathfrak{M}, w \Vdash \theta$. (b) $\mathfrak{M}, w \Vdash \chi$. Then $\mathfrak{M}, w \Vdash \Delta_2 \cup \{\chi\}$. Since $\Delta_2 \cup \chi \vDash \theta$, we have $\mathfrak{M}, w \Vdash \theta$. So in either case, $\mathfrak{M}, w \Vdash \theta$, as we wanted to show.

6. The derivation ends with $\rightarrow$Intro concluding $\psi \rightarrow \chi$. Then the premise is χ, and the derivation ending in the premise has undischarged assumptions $\Gamma \cup \{\psi\}$. So we have that $\Gamma \cup \{\psi\} \vdash \chi$, and by induction hypothesis that $\Gamma \cup \{\psi\} \vDash \chi$. We have to show that $\Gamma \vDash \psi \rightarrow \chi$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma$. We want to show that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \psi$, then $\mathfrak{M}, w' \Vdash \chi$. So assume that Rww' and $\mathfrak{M}, w' \Vdash \psi$. By **??**, $\mathfrak{M}, w' \Vdash \Gamma$. Since $\Gamma \cup \{\psi\} \vDash \chi$, $\mathfrak{M}, w' \Vdash \chi$, which is what we wanted to show.

7. The derivation ends in $\rightarrow$Elim and conclusion χ. The premises are $\psi \rightarrow \chi$ and ψ, with derivations from undischarged assumptions Γ, Δ. So we have $\Gamma \vdash \psi \rightarrow \chi$ and $\Delta \vdash \psi$. By inductive hypothesis, $\Gamma \vDash \psi \rightarrow \chi$ and $\Delta \vDash \psi$. We have to show that $\Gamma \cup \Delta \vDash \chi$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \psi \rightarrow \chi$, $\mathfrak{M}, w \Vdash \psi \rightarrow \chi$. By definition, this means that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \psi$ then $\mathfrak{M}, w' \Vdash \chi$. Since R is reflexive, w is among the w' such that Rww', i.e., we have that if $\mathfrak{M}, w \Vdash \psi$ then $\mathfrak{M}, w \Vdash \chi$. Since $\mathfrak{M}, w \Vdash \Delta$ and $\Delta \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. So, $\mathfrak{M}, w \Vdash \chi$, as we wanted to show.

8. The derivation ends in $\bot_I$, concluding φ. The premise is $\bot$ and the undischarged assumptions of the derivation of the premise are Γ. Then $\Gamma \vdash \bot$. By inductive hypothesis, $\Gamma \vDash \bot$. We have to show $\Gamma \vDash \varphi$.

   We proceed indirectly. If $\Gamma \nvDash \varphi$ there is a model $\mathfrak{M}$ and world w such that $\mathfrak{M}, w \Vdash \Gamma$ and $\mathfrak{M}, w \nVdash \varphi$. Since $\Gamma \vDash \bot$, $\mathfrak{M}, w \Vdash \bot$. But that's impossible, since by definition, $\mathfrak{M}, w \nVdash \bot$. So $\Gamma \vDash \varphi$.

9. The derivation ends in $\neg$Intro: Exercise.

10. The derivation ends in $\neg$Elim: Exercise. □

Problem sc.1. Complete the proof of Theorem sc.1. For the cases for $\neg$Intro and $\neg$Elim, use the definition of $\mathfrak{M}, w \Vdash \neg\varphi$ in **??**, i.e., don't treat $\neg\varphi$ as defined by $\varphi \rightarrow \bot$.

Problem sc.2. Show that the following formulas are not derivable in intuitionistic logic:

1. $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$
2. $(\neg\neg\varphi \rightarrow \varphi) \rightarrow (\varphi \vee \neg\varphi)$
3. $(\varphi \rightarrow \psi \vee \chi) \rightarrow \big((\varphi \rightarrow \psi) \vee (\varphi \rightarrow \chi)\big)$

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Bibliography