Chapter udf

Soundness and Completeness

This chapter collects soundness and completeness results for propositional intuitionistic logic. It needs an introduction. The completeness proof makes use of facts about provability that should be stated and proved explicitly somehwere.

sc.1 Soundness of Axiomatic Derivations

int:sc:sax:
sec

The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, e.g., in the Semantics chapter.

int:sc:sax:
thm:soundness

Theorem sc.1 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*

Proof. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the number n of formulas in the derivation of φ from Γ. We show that if φ_1, …, $\varphi_n = \varphi$ is a derivation from Γ, then $\Gamma \vDash \varphi_n$. Note that if φ_1, …, φ_n is a derivation, so is φ_1, …, φ_k for any $k < n$.

There are no derivations of length 0, so for $n = 0$ the claim holds vacuously. So the claim holds for all derivations of length $< n$. We distinguish cases according to the justification of φ_n.

1. φ_n is an axiom. All axioms are valid, so $\Gamma \vDash \varphi_n$ for any Γ.

2. $\varphi_n \in \Gamma$. Then for any $\mathfrak{M}$ and w, if $\mathfrak{M}, w \Vdash \Gamma$, obviously $\mathfrak{M} \Vdash \Gamma\varphi_n[w]$, i.e., $\Gamma \vDash \varphi$.

3. φ_n follows by MP from φ_i and $\varphi_j \equiv \varphi_i \to \varphi_n$. φ_1, …, φ_i and φ_1, …, φ_j are derivations from Γ, so by inductive hypothesis, $\Gamma \vDash \varphi_i$ and $\Gamma \vDash \varphi_i \to \varphi_n$.

Suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \varphi_i \to \varphi_n$, $\mathfrak{M}, w \Vdash \varphi_i \to \varphi_n$. By definition, this means that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \varphi_i$ then $\mathfrak{M}, w' \Vdash \varphi_n$. Since R is reflexive, w is among the w' such that Rww', i.e., we have that if $\mathfrak{M}, w \Vdash \varphi_i$ then $\mathfrak{M}, w \Vdash \varphi_n$. Since $\Gamma \vDash \varphi_i$, $\mathfrak{M}, w \Vdash \varphi_i$. So, $\mathfrak{M}, w \Vdash \varphi_n$, as we wanted to show.

□

sc.2 Soundness of Natural Deduction

int:sc:snd:sec

Theorem sc.2 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.* *int:sc:snd:thm:soundness*

Proof. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the derivation of φ from Γ.

1. If the derivation consists of just the assumption φ, we have $\varphi \vdash \varphi$, and want to show that $\varphi \vDash \varphi$. Consider any model $\mathfrak{M}$ such that $\mathfrak{M} \Vdash \varphi$. Then trivially $\mathfrak{M} \Vdash \varphi$.

2. The derivation ends in $\wedge$Intro: The derivations of the premises ψ from undischarged assumptions Γ and of χ from undischarged assumptions Δ show that $\Gamma \vdash \psi$ and $\Delta \vdash \chi$. By induction hypothesis we have that $\Gamma \vDash \psi$ and $\Gamma \vDash \chi$. We have to show that $\Gamma \cup \Delta \vDash \varphi \wedge \psi$, since the undischarged assumptions of the entire derivation are Γ together with Δ. So suppose $\mathfrak{M} \Vdash \Gamma \cup \Delta$. Then also $\mathfrak{M} \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M} \Vdash \psi$. Similarly, $\mathfrak{M} \Vdash \chi$. So $\mathfrak{M} \Vdash \psi \wedge \chi$.

3. The derivation ends in $\wedge$Elim: The derivation of the premise $\psi \wedge \chi$ from undischarged assumptions Γ shows that $\Gamma \vdash \psi \wedge \chi$. By induction hypothesis, $\Gamma \vDash \psi \wedge \chi$. We have to show that $\Gamma \vDash \psi$. So suppose $\mathfrak{d} \Vdash$. Since $\Gamma \vDash \psi \wedge \chi$, $\mathfrak{M} \Vdash \psi \wedge \chi$. Then also $\mathfrak{M} \Vdash \psi$. Similarly if $\wedge$Elim ends in χ, then $\Gamma \vDash \chi$.

4. The derivation ends in $\vee$Intro: Suppose the premise is ψ, and the undischarged assumptions of the derivation ending in ψ are Γ. Then we have $\Gamma \vdash \psi$ and by inductive hypothesis, $\Gamma \vDash B$. We have to show that $\Gamma \vDash \psi \vee \chi$. Suppose $\mathfrak{M} \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M} \Vdash \psi$. But then also $\mathfrak{M} \Vdash \psi \vee \chi$. Similarly, if the premise is χ, we have that $\Gamma \vDash \chi$.

5. The derivation ends in $\vee$Elim: The derivations ending in the premises are of $\psi \vee \chi$ from undischarged assumptions Γ, of θ from undischarged assumptions $\Delta_1 \cup \{\psi\}$, and of θ from undischarged assumptions $\Delta_2 \cup \{\chi\}$. So we have $\Gamma \vdash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vdash \theta$, and $\Delta_2 \cup \{\chi\} \vdash \theta$. By induction hypothesis, $\Gamma \vDash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vDash \theta$, and $\Delta_2 \cup \{\chi\} \vDash \theta$. We have to prove that $\Gamma \cup \Delta_1 \cup \Delta_2 \vDash \theta$.

Suppose $\mathfrak{M} \Vdash \Gamma \cup \Delta_1 \cup \Delta_2$. Then $\mathfrak{M} \Vdash \Gamma$ and since $\Gamma \vDash \psi \lor \chi$, $\mathfrak{M} \Vdash \psi \lor \chi$. By definition of $\mathfrak{M} \Vdash$, either $\mathfrak{M} \Vdash \psi$ or $\mathfrak{M} \Vdash \chi$. So we distinguish cases: (a) $\mathfrak{M} \Vdash \psi$. Then $\mathfrak{M} \Vdash \Delta_1 \cup \{\psi\}$. Since $\Delta_1 \cup \psi \vDash \theta$, we have $\mathfrak{M} \Vdash \theta$. (b) $\mathfrak{M} \Vdash \chi$. Then $\mathfrak{M} \Vdash \Delta_2 \cup \{\chi\}$. Since $\Delta_2 \cup \chi \vDash \theta$, we have $\mathfrak{M} \Vdash \theta$. So in either case, $\mathfrak{M} \Vdash \theta$, as we wanted to show.

6. The derivation ends with $\rightarrow$Intro concluding $\psi \rightarrow \chi$. Then the premise is χ, and the derivation ending in the premise has undischarged assumptions $\Gamma \cup \{\psi\}$. So we have that $\Gamma \cup \{\psi\} \vdash \chi$, and by induction hypothesis that $\Gamma \cup \{\psi\} \vDash \chi$. We have to show that $\Gamma \vDash \psi \rightarrow \chi$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma$. We want to show that that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \psi$, then $\mathfrak{M}, w' \Vdash \chi$. So assume that Rww' and $\mathfrak{M}, w' \Vdash \psi$. By **??**, $\mathfrak{M}, w' \Vdash \Gamma$. Since $\Gamma \cup \{\psi\} \vDash \chi$, $\mathfrak{M}, w' \Vdash \chi$, which is what we wanted to show.

7. The derivation ends in $\rightarrow$Elim and conclusion χ. The premises are $\psi \rightarrow \chi$ amd ψ, with derivations from undischarged assumptions Γ, Δ. So we have $\Gamma \vdash \psi \rightarrow \chi$ and $\Delta \vdash \psi$. By inductive hypothesis, $\Gamma \vDash \psi \rightarrow \chi$ and $\Delta \vDash \psi$. We have to show that $\Gamma \cup \Delta \vDash \chi$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \psi \rightarrow \chi$, $\mathfrak{M}, w \Vdash \psi \rightarrow \chi$. By definition, this means that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \psi$ then $\mathfrak{M}, w' \Vdash \chi$. Since R is reflexive, w is among the w' such that Rww', i.e., we have that if $\mathfrak{M}, w \Vdash \psi$ then $\mathfrak{M}, w \Vdash \chi$. Since $\mathfrak{M}, w \Vdash \Delta$ and $\Delta \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. So, $\mathfrak{M}, w \Vdash \chi$, as we wanted to show.

8. The derivation ends in $\bot_I$, concluding φ. The premise is $\bot$ and the undischarged assumptions of the derivation of the premise are Γ. Then $\Gamma \vdash \bot$. By inductive hypothesis, $\Gamma \vDash \bot$. We have to show $\Gamma \vDash \varphi$.

   We proceed indirectly. If $\Gamma \nvDash \varphi$ there is a model $\mathfrak{M}$ and world w such that $\mathfrak{M}, w \Vdash \Gamma$ and $\mathfrak{M}, w \nVdash \varphi$. Since $\Gamma \vDash \bot$, $\mathfrak{M}, w \Vdash \bot$. But that's impossible, since by definition, $\mathfrak{M}, w \nVdash \bot$. So $\Gamma \vDash \varphi$.

9. The derivation endss in $\neg$Intro: Exercise.

10. The derivation ends in $\neg$Elim: Exercise.

□

Problem sc.1. Complete the proof of Theorem sc.2. For the cases for $\neg$Intro and $\neg$Elim, use the definition of $\mathfrak{M}, w \Vdash \neg\varphi$ in **??**, i.e., don't treat $\neg\varphi$ as defined by $\varphi \rightarrow \bot$.

sc.3 Lindenbaum's Lemma

int:sc:lin: sec

int:sc:lin: defn:prime **Definition sc.3.** A set of formulas Γ is *prime* iff

1. Γ is consistent. int:sc:lin: defn:prime1

2. If $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$, and int:sc:lin: defn:prime2

3. If $\varphi \lor \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$. int:sc:lin: defn:prime3

Lemma sc.4 (Lindenbaum's Lemma)**.** *If $\Gamma \nvdash \varphi$, there is a $\Gamma^* \supseteq \Gamma$ such that Γ^* is prime and $\Gamma^* \nvdash \varphi$.* *int:sc:lin: lem:lindenbaum*

Proof. Let $\psi_1 \lor \chi_1$, $\psi_2 \lor \chi_2$, …, be an enumeration of all formulas of the form $\psi \lor \chi$. We'll define an increasing sequence of sets of formulas Γ_n, where each Γ_{n+1} is defined as Γ_n together with one new formula. Γ^* will be the union of all Γ_n. The new formulas are selected so as to ensure that Γ^* is prime and still $\Gamma^* \nvdash \varphi$. This means that at each step we should find the first disjunction $\psi_i \lor \chi_i$ such that:

1. $\Gamma_n \vdash \psi_i \lor \chi_i$

2. $\psi_i \notin \Gamma_n$ and $\chi_i \notin \Gamma_n$

We add to Γ_n either ψ_i if $\Gamma_n \cup \{\psi_i\} \nvdash \varphi$, or χ_i otherwise. We'll have to show that this works. For now, let's define $i(n)$ as the least i such that (1) and (2) hold.

Define $\Gamma_0 = \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_{i(n)}\} & \text{if } \Gamma_n \cup \{\psi_{i(n)}\} \nvdash \varphi \\ \Gamma_n \cup \{\chi_{i(n)}\} & \text{otherwise} \end{cases}$$

If $i(n)$ is undefined, i.e., whenever $\Gamma \vdash \psi \lor \chi$, either $\psi \in \gamma_n$ or $\chi \in \Gamma_n$, we let $\Gamma_{n+1} = \Gamma_n$. Now let $\Gamma^* = \bigcup_{n=0}^{\infty} \Gamma_n$

First we show that for all n, $\Gamma_n \nvdash \varphi$. We proceed by induction on n. For $n = 0$ the claim holds by the hypothesis of the theorem, i.e., $\Gamma \nvdash \varphi$. If $n > 0$, we have to show that if $\Gamma_n \nvdash \varphi$ then $\Gamma_{n+1} \nvdash \varphi$. If $i(n)$ is undefined, $\Gamma_{n+1} = \Gamma_n$ and there is nothing to prove. So suppose $i(n)$ is defined. For simplicity, let $i = i(n)$.

We'll prove the contrapositive of the claim. Suppose $\Gamma_{n+1} \vdash \varphi$. By construction, $\Gamma_{n+1} = \Gamma_n \cup \{\psi_i\}$ if $\Gamma_n \cup \{\psi_i\} \nvdash \varphi$, or else $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. It clearly can't be the first, since then $\Gamma_{n+1} \nvdash \varphi$. Hence, $\Gamma_n \cup \{\psi_i\} \vdash \varphi$ and $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. By definition of $i(n)$, we have that $\Gamma_n \vdash \psi_i \lor \chi_i$. We have $\Gamma_n \cup \{\psi_i\} \vdash \varphi$. We also have $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\} \vdash \varphi$. Hence, $\Gamma_n \vdash \varphi$, which is what we wanted to show.

If $\Gamma^* \vdash \varphi$, there would be some finite subset $\Gamma' \subseteq \Gamma^*$ such that $\Gamma' \vdash \varphi$. Each $\theta \in \Gamma'$ must be in Γ_i for some i. Let n be the largest of these. Since $\Gamma_i \subseteq \Gamma_n$ if $i \leq n$, $\Gamma' \subseteq \Gamma_n$. But then $\Gamma_n \vdash \varphi$, contrary to our proof above that $\Gamma_n \nvdash \varphi$.

Lastly, we show that Γ^* is prime, i.e., satisfies conditions (1), (2), and (3) of Definition sc.3.

First, $\Gamma^* \nvdash \varphi$, so Γ^* is consistent, so (1) holds.

We now show that if $\Gamma^* \vdash \psi \lor \chi$, then either $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$. This proves (3), since if $\psi \in \Gamma^*$ then also $\Gamma^* \vdash \psi$, and similarly for χ. So assume $\Gamma^* \vdash \psi \lor \chi$ but $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$. Since $\Gamma^* \vdash \psi \lor \chi$, $\Gamma_n \vdash \psi \lor \chi$ for some n. $\psi \lor \chi$ appears on the enumeration of all disjunctions, say as $\psi_j \lor \chi_j$. $\psi_j \lor \chi_j$ satisfies the properties in the definition of $i(n)$, namely we have $\Gamma_n \vdash \psi_j \lor \chi_j$, while $\psi_j \notin \Gamma_n$ and $\chi_j \notin \Gamma_n$. At each stage, at least one fewer disjunction $\psi_i \lor \chi_i$ satisfies the conditions (since at each stage we add either ψ_i or χ_i), so at some stage m we will have $j = i(\Gamma_m)$. But then either $\psi \in \Gamma_{m+1}$ or $\chi \in \Gamma_{m+1}$, contrary to the assumption that $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$.

Now suppose $\Gamma^* \vdash \varphi$. Then $\Gamma^* \vdash \varphi \lor \varphi$. But we've just proved that if $\Gamma^* \vdash \varphi \lor \varphi$ then $\varphi \in \Gamma^*$. Hence, Γ^* satisfies (1) of Definition sc.3. □

sc.4 The Canonical Model

int:sc:mod:sec The worls in our model will be finite sequences σ of natural numbers, i.e., $\sigma \in \mathbb{N}^*$. Note that $\mathbb{N}^*$ is inductively defined by:

1. $\Lambda \in \mathbb{N}^*$.
2. If $\sigma \in \mathbb{N}^*$ and $n \in \Sigma$, then $\sigma.n \in \mathbb{N}^*$ (where $\sigma.n$ is $\sigma \frown \langle n \rangle$).
3. Nothing else is in $\mathbb{N}^*$.

So we can use $\mathbb{N}^*$ to give inductive definitions.

Let $\langle \psi_1, \chi_1 \rangle$, $\langle \psi_2, \chi_s \rangle$, …, be an enumeration of all pairs of formulas. Given a set of formulas Δ, define $\Delta(\sigma)$ by induction as follows:

1. $\Delta(\Lambda) = \Delta$
2. $\Delta(\sigma.n) =$
$$\begin{cases} (\Delta(\sigma) \cup \{\psi_n\})^* & \text{if } \Delta(\sigma) \cup \{\psi_n\} \nvdash \chi_n \\ \Delta(\sigma) & \text{otherwise} \end{cases}$$

Here by $(\Delta(\sigma) \cup \{\psi_n\})^*$ we mean the prime set of formulas which exists by Lemma sc.4 applied to the set $\Delta(\sigma) \cup \{\psi_n\}$. Note that by this definition, if $\Delta(\sigma) \cup \{\psi_n\} \nvdash \chi_n$, then $\Delta(\sigma.n) \vdash \psi_n$ and $\Delta(\sigma.n) \nvdash \chi_n$. Note also that $\Delta(\sigma) \subseteq \Delta(\sigma.n)$ for any n. If Δ is prime, then $\Delta(\sigma)$ is prime for all σ.

int:sc:mod:defn:canonical-model **Definition sc.5.** Suppose Δ is prime. Then the *canonical model* for Δ is defined by:

1. $W = \mathbb{N}^*$, the set of finite sequences of natural numbers.
2. R is the partial order according to which $R\sigma\sigma'$ iff σ is an initial segment of σ' (i.e., $\sigma' = \sigma \frown \sigma''$ for some sequence σ'').
3. $V(p) = \{\sigma : p \in \Delta(\sigma)\}$.

It is easy to verify that R is indeed a partial order. Also, the monotonicity condition on V is satisfied. Since $\Delta(\sigma) \subseteq \Delta(\sigma.n)$ we get $\Delta(\sigma) \subseteq \Delta(\sigma')$ whenever $R\sigma\sigma'$ by induction on σ.

sc.5 The Truth Lemma

int:sc:tru:sec

Lemma sc.6. *If Δ is prime, then $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\Delta(\sigma) \vdash \varphi$.* int:sc:tru:lem:truth

Proof. By induction on φ.

1. $\varphi \equiv \bot$: Since $\Delta(\sigma)$ is prime, it is consistent, so $\Delta(\sigma) \nvdash \varphi$. By definition, $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

2. $\varphi \equiv p$: By definition of $\Vdash$, $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\sigma \in V(p)$, i.e., $\Delta(\sigma) \vdash \varphi$.

3. $\varphi \equiv \neg\psi$: exercise.

4. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ iff $\Delta(\sigma) \vdash \psi$, and similarly for χ. But $\Delta(\sigma) \vdash \psi$ and $\Delta(\sigma) \vdash \chi$ iff $\Delta(\sigma) \vdash \varphi$.

5. $\varphi \equiv \psi \vee \chi$: $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. By induction hypothesis, this holds iff $\Delta(\sigma) \vdash \psi$ of $\Delta(\sigma) \vdash \chi$. We have to show that this in turn holds iff $\Delta(\sigma) \vdash \varphi$. The left-to-right direction is clear. The right-to-left direction follows since $\Delta(\sigma)$ is prime.

6. $\varphi \equiv \psi \rightarrow \chi$: First the contrapositive of the left-to-right direction: Assume $\Delta(\sigma) \nvdash \psi \rightarrow \chi$. Then also $\Gamma * (\sigma) \cup \{\psi\} \nvdash \chi$. Since $\langle \psi, \chi \rangle$ is $\langle \psi_n, \chi_n \rangle$ for some n, we have $\Delta(\sigma.n) = (\Delta(\sigma) \cup \{\psi\})^*$, and $\Delta(\sigma.n) \vdash \psi$ but $\nvdash \chi$. By inductive hypothesis, $\mathfrak{M}(\Delta), \sigma.n \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma.n \nVdash \chi$. Since $R\sigma(\sigma.n)$, this means that $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   Now assume $\Delta(\sigma) \vdash \psi \rightarrow \chi$, and let $R\sigma\sigma'$. Since $\Delta(\sigma) \subseteq \Delta(\sigma')$, we have: if $\Delta(\sigma') \vdash \psi$, then $\Delta(\sigma') \vdash \chi$. In other words, for every σ' such that $R\sigma\sigma'$, either $\Delta(\sigma') \nvdash \psi$ or $\Delta(\sigma') \vdash \chi$. By induction hypothesis, this means that whenever $R\sigma\sigma'$, either $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma' \Vdash \chi$, i.e., $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

□

sc.6 The Completeness Theorem

int:sc:cpl:sec

Theorem sc.7. *If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.* int:sc:cpl:thm:completeness

Proof. We prove the contrapositive: Suppose $\Gamma \nvdash \varphi$. Then by Lemma sc.4, there is a prime set $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \nvdash \varphi$. Consider the canonical model $\mathfrak{M}(\Gamma^*)$ for Γ^* as defined in Definition sc.5. For any $\psi \in \Gamma$, $\Gamma^* \vdash \psi$. Note that $\Gamma * (\Lambda) = \Gamma *$. By the Truth Lemma (Lemma sc.6), we have $\mathfrak{M}(\Gamma^*), \Lambda \Vdash \psi$ for all $\psi \in \Gamma$ and $\mathfrak{M}(\Gamma^*), \Lambda \nVdash \varphi$. This shows that $\Gamma \nvDash \varphi$. □

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