The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, e.g., in the Semantics chapter.

**Theorem sc.1 (Soundness).** If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

**Proof.** We prove that if $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$. The proof is by induction on the number $n$ of formulas in the derivation of $\varphi$ from $\Gamma$. We show that if $\varphi_1, \ldots, \varphi_n = \varphi$ is a derivation from $\Gamma$, then $\Gamma \models \varphi_n$. Note that if $\varphi_1, \ldots, \varphi_n$ is a derivation, so is $\varphi_1, \ldots, \varphi_k$ for any $k < n$.

There are no derivations of length 0, so for $n = 0$ the claim holds vacuously. So the claim holds for all derivations of length $< n$. We distinguish cases according to the justification of $\varphi_n$.

1. $\varphi_n$ is an axiom. All axioms are valid, so $\Gamma \models \varphi_n$ for any $\Gamma$.
2. $\varphi_n \in \Gamma$. Then for any $\mathcal{M}$ and $w$, if $\mathcal{M}, w \models \Gamma$, obviously $\mathcal{M} \models \Gamma \varphi_n[w]$, i.e., $\Gamma \models \varphi$.
3. $\varphi_n$ follows by $mp$ from $\varphi_i$ and $\varphi_j \equiv \varphi_i \rightarrow \varphi_n$. $\varphi_1, \ldots, \varphi_i$ and $\varphi_1, \ldots, \varphi_j$ are derivations from $\Gamma$, so by inductive hypothesis, $\Gamma \models \varphi_i$ and $\Gamma \models \varphi_i \rightarrow \varphi_n$.

Suppose $\mathcal{M}, w \models \Gamma$. Since $\mathcal{M}, w \models \Gamma$ and $\Gamma \models \varphi_i \rightarrow \varphi_n$, $\mathcal{M}, w \models \varphi_i \rightarrow \varphi_n$. By definition, this means that for all $w'$ such that $Rww'$, if $\mathcal{M}, w' \models \varphi_i$, then $\mathcal{M}, w' \models \varphi_n$. Since $R$ is reflexive, $w$ is among the $w'$ such that $Rww'$, i.e., we have that if $\mathcal{M}, w \models \varphi_i$ then $\mathcal{M}, w \models \varphi_n$. Since $\Gamma \models \varphi_i$, $\mathcal{M}, w \models \varphi_i$. So, $\mathcal{M}, w \models \varphi_n$, as we wanted to show. □

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**Bibliography**