

## sc.1 Soundness of Axiomatic Derivations

int:sc:sax:  
sec

The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, e.g., in the Semantics chapter.

int:sc:sax:  
thm:soundness

**Theorem sc.1** (Soundness). *If  $\Gamma \vdash \varphi$ , then  $\Gamma \vDash \varphi$ .*

*Proof.* We prove that if  $\Gamma \vdash \varphi$ , then  $\Gamma \vDash \varphi$ . The proof is by induction on the number  $n$  of **formulas** in the **derivation** of  $\varphi$  from  $\Gamma$ . We show that if  $\varphi_1, \dots, \varphi_n = \varphi$  is a **derivation** from  $\Gamma$ , then  $\Gamma \vDash \varphi_n$ . Note that if  $\varphi_1, \dots, \varphi_n$  is a **derivation**, so is  $\varphi_1, \dots, \varphi_k$  for any  $k < n$ .

There are no **derivations** of length 0, so for  $n = 0$  the claim holds vacuously. So the claim holds for all **derivations** of length  $< n$ . We distinguish cases according to the justification of  $\varphi_n$ .

1.  $\varphi_n$  is an axiom. All axioms are valid, so  $\Gamma \vDash \varphi_n$  for any  $\Gamma$ .
2.  $\varphi_n \in \Gamma$ . Then for any  $\mathfrak{M}$  and  $w$ , if  $\mathfrak{M}, w \Vdash \Gamma$ , obviously  $\mathfrak{M} \Vdash \Gamma \varphi_n[w]$ , i.e.,  $\Gamma \vDash \varphi$ .
3.  $\varphi_n$  follows by MP from  $\varphi_i$  and  $\varphi_j \equiv \varphi_i \rightarrow \varphi_n$ .  $\varphi_1, \dots, \varphi_i$  and  $\varphi_1, \dots, \varphi_j$  are **derivations** from  $\Gamma$ , so by inductive hypothesis,  $\Gamma \vDash \varphi_i$  and  $\Gamma \vDash \varphi_i \rightarrow \varphi_n$ .

Suppose  $\mathfrak{M}, w \Vdash \Gamma$ . Since  $\mathfrak{M}, w \Vdash \Gamma$  and  $\Gamma \vDash \varphi_i \rightarrow \varphi_n$ ,  $\mathfrak{M}, w \Vdash \varphi_i \rightarrow \varphi_n$ . By definition, this means that for all  $w'$  such that  $Rww'$ , if  $\mathfrak{M}, w' \Vdash \varphi_i$  then  $\mathfrak{M}, w' \Vdash \varphi_n$ . Since  $R$  is reflexive,  $w$  is among the  $w'$  such that  $Rww'$ , i.e., we have that if  $\mathfrak{M}, w \Vdash \varphi_i$  then  $\mathfrak{M}, w \Vdash \varphi_n$ . Since  $\Gamma \vDash \varphi_i$ ,  $\mathfrak{M}, w \Vdash \varphi_i$ . So,  $\mathfrak{M}, w \Vdash \varphi_n$ , as we wanted to show.

□

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## Bibliography