

sc.1 Soundness of Axiomatic Derivations

int:sc:sax:
sec

The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, e.g., in the Semantics chapter.

int:sc:sax:
thm:soundness

Theorem sc.1 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*

Proof. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the number n of **formulas** in the **derivation** of φ from Γ . We show that if $\varphi_1, \dots, \varphi_n = \varphi$ is a **derivation** from Γ , then $\Gamma \vDash \varphi_n$. Note that if $\varphi_1, \dots, \varphi_n$ is a **derivation**, so is $\varphi_1, \dots, \varphi_k$ for any $k < n$.

There are no **derivations** of length 0, so for $n = 0$ the claim holds vacuously. So the claim holds for all **derivations** of length $< n$. We distinguish cases according to the justification of φ_n .

1. φ_n is an axiom. All axioms are valid, so $\Gamma \vDash \varphi_n$ for any Γ .
2. $\varphi_n \in \Gamma$. Then for any \mathfrak{M} and w , if $\mathfrak{M}, w \Vdash \Gamma$, obviously $\mathfrak{M} \Vdash \Gamma \varphi_n[w]$, i.e., $\Gamma \vDash \varphi$.
3. φ_n follows by MP from φ_i and $\varphi_j \equiv \varphi_i \rightarrow \varphi_n$. $\varphi_1, \dots, \varphi_i$ and $\varphi_1, \dots, \varphi_j$ are **derivations** from Γ , so by inductive hypothesis, $\Gamma \vDash \varphi_i$ and $\Gamma \vDash \varphi_i \rightarrow \varphi_n$.

Suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \varphi_i \rightarrow \varphi_n$, $\mathfrak{M}, w \Vdash \varphi_i \rightarrow \varphi_n$. By definition, this means that for all w' such that Rww' , if $\mathfrak{M}, w' \Vdash \varphi_i$ then $\mathfrak{M}, w' \Vdash \varphi_n$. Since R is reflexive, w is among the w' such that Rww' , i.e., we have that if $\mathfrak{M}, w \Vdash \varphi_i$ then $\mathfrak{M}, w \Vdash \varphi_n$. Since $\Gamma \vDash \varphi_i$, $\mathfrak{M}, w \Vdash \varphi_i$. So, $\mathfrak{M}, w \Vdash \varphi_n$, as we wanted to show. \square

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Bibliography