sc.1 Soundness of Axiomatic Derivations

The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, e.g., in the Semantics chapter.

**Theorem sc.1 (Soundness).** If \( \Gamma \vdash \varphi \), then \( \Gamma \models \varphi \).

**Proof.** We prove that if \( \Gamma \vdash \varphi \), then \( \Gamma \models \varphi \). The proof is by induction on the number \( n \) of formulas in the derivation of \( \varphi \) from \( \Gamma \). We show that if \( \varphi_1, \ldots, \varphi_n = \varphi \) is a derivation from \( \Gamma \), then \( \Gamma \models \varphi_n \). Note that if \( \varphi_1, \ldots, \varphi_n \) is a derivation, so is \( \varphi_1, \ldots, \varphi_k \) for any \( k < n \).

There are no derivations of length 0, so for \( n = 0 \) the claim holds vacuously. So the claim holds for all derivations of length \( < n \). We distinguish cases according to the justification of \( \varphi_n \).

1. \( \varphi_n \) is an axiom. All axioms are valid, so \( \Gamma \models \varphi_n \) for any \( \Gamma \).

2. \( \varphi_n \in \Gamma \). Then for any \( M \) and \( w \), if \( M, w \models \Gamma \), obviously \( M \models \Gamma \varphi_n[w] \), i.e., \( \Gamma \models \varphi \).

3. \( \varphi_n \) follows by mp from \( \varphi_i \) and \( \varphi_j \equiv \varphi_i \rightarrow \varphi_n \). \( \varphi_1, \ldots, \varphi_i \) and \( \varphi_1, \ldots, \varphi_j \), \( \ldots, \varphi_n \) are derivations from \( \Gamma \), so by inductive hypothesis, \( \Gamma \models \varphi_i \) and \( \Gamma \models \varphi_i \rightarrow \varphi_n \).

   Suppose \( M, w \models \Gamma \). Since \( M, w \models \Gamma \) and \( \Gamma \models \varphi_i \rightarrow \varphi_n \), \( M, w \models \varphi_i \rightarrow \varphi_n \).

   By definition, this means that for all \( w' \) such that \( Rww' \), if \( M, w' \models \varphi_i \) then \( M, w' \models \varphi_n \). Since \( R \) is reflexive, \( w \) is among the \( w' \) such that \( Rww' \), i.e., we have that if \( M, w \models \varphi_i \) then \( M, w \models \varphi_n \). Since \( \Gamma \models \varphi_i \), \( M, w \models \varphi_i \.

   So, \( M, w \models \varphi_n \), as we wanted to show.

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Bibliography