

sc.1 Lindenbaum's Lemma

int:sc:lin:
sec

int:sc:lin: **Definition sc.1.** A set of formulas Γ is *prime* iff

defn:prime

int:sc:lin:

1. Γ is consistent.

defn:prime1

int:sc:lin:

2. If $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$, and

defn:prime2

int:sc:lin:

3. If $\varphi \vee \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

defn:prime3

int:sc:lin:
lem:lindenbaum

Lemma sc.2 (Lindenbaum's Lemma). *If $\Gamma \not\vdash \varphi$, there is a $\Gamma^* \supseteq \Gamma$ such that Γ^* is prime and $\Gamma^* \not\vdash \varphi$.*

Proof. Let $\psi_1 \vee \chi_1, \psi_2 \vee \chi_2, \dots$, be an enumeration of all formulas of the form $\psi \vee \chi$. We'll define an increasing sequence of sets of formulas Γ_n , where each Γ_{n+1} is defined as Γ_n together with one new formula. Γ^* will be the union of all Γ_n . The new formulas are selected so as to ensure that Γ^* is prime and still $\Gamma^* \not\vdash \varphi$. This means that at each step we should find the first disjunction $\psi_i \vee \chi_i$ such that:

1. $\Gamma_n \vdash \psi_i \vee \chi_i$
2. $\psi_i \notin \Gamma_n$ and $\chi_i \notin \Gamma_n$

We add to Γ_n either ψ_i if $\Gamma_n \cup \{\psi_i\} \not\vdash \varphi$, or χ_i otherwise. We'll have to show that this works. For now, let's define $i(n)$ as the least i such that (1) and (2) hold.

Define $\Gamma_0 = \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_{i(n)}\} & \text{if } \Gamma_n \cup \{\psi_{i(n)}\} \not\vdash \varphi \\ \Gamma_n \cup \{\chi_{i(n)}\} & \text{otherwise} \end{cases}$$

If $i(n)$ is undefined, i.e., whenever $\Gamma \vdash \psi \vee \chi$, either $\psi \in \Gamma_n$ or $\chi \in \Gamma_n$, we let $\Gamma_{n+1} = \Gamma_n$. Now let $\Gamma^* = \bigcup_{n=0}^{\infty} \Gamma_n$

First we show that for all n , $\Gamma_n \not\vdash \varphi$. We proceed by induction on n . For $n = 0$ the claim holds by the hypothesis of the theorem, i.e., $\Gamma \not\vdash \varphi$. If $n > 0$, we have to show that if $\Gamma_n \not\vdash \varphi$ then $\Gamma_{n+1} \not\vdash \varphi$. If $i(n)$ is undefined, $\Gamma_{n+1} = \Gamma_n$ and there is nothing to prove. So suppose $i(n)$ is defined. For simplicity, let $i = i(n)$.

We'll prove the contrapositive of the claim. Suppose $\Gamma_{n+1} \vdash \varphi$. By construction, $\Gamma_{n+1} = \Gamma_n \cup \{\psi_i\}$ if $\Gamma_n \cup \{\psi_i\} \not\vdash \varphi$, or else $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. It clearly can't be the first, since then $\Gamma_{n+1} \not\vdash \varphi$. Hence, $\Gamma_n \cup \{\psi_i\} \vdash \varphi$ and $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. By definition of $i(n)$, we have that $\Gamma_n \vdash \psi_i \vee \chi_i$. We have $\Gamma_n \cup \{\psi_i\} \vdash \varphi$. We also have $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\} \vdash \varphi$. Hence, $\Gamma_n \vdash \varphi$, which is what we wanted to show.

If $\Gamma^* \vdash \varphi$, there would be some finite subset $\Gamma' \subseteq \Gamma^*$ such that $\Gamma' \vdash \varphi$. Each $\theta \in \Gamma'$ must be in Γ_i for some i . Let n be the largest of these. Since

$\Gamma_i \subseteq \Gamma_n$ if $i \leq n$, $\Gamma' \subseteq \Gamma_n$. But then $\Gamma_n \vdash \varphi$, contrary to our proof above that $\Gamma_n \not\vdash \varphi$.

Lastly, we show that Γ^* is prime, i.e., satisfies conditions (1), (2), and (3) of [Definition sc.1](#).

First, $\Gamma^* \not\vdash \varphi$, so Γ^* is consistent, so (1) holds.

We now show that if $\Gamma^* \vdash \psi \vee \chi$, then either $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$. This proves (3), since if $\psi \in \Gamma^*$ then also $\Gamma^* \vdash \psi$, and similarly for χ . So assume $\Gamma^* \vdash \psi \vee \chi$ but $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$. Since $\Gamma^* \vdash \psi \vee \chi$, $\Gamma_n \vdash \psi \vee \chi$ for some n . $\psi \vee \chi$ appears on the enumeration of all disjunctions, say as $\psi_j \vee \chi_j$. $\psi_j \vee \chi_j$ satisfies the properties in the definition of $i(n)$, namely we have $\Gamma_n \vdash \psi_j \vee \chi_j$, while $\psi_j \notin \Gamma_n$ and $\chi_j \notin \Gamma_n$. At each stage, at least one fewer disjunction $\psi_i \vee \chi_i$ satisfies the conditions (since at each stage we add either ψ_i or χ_i), so at some stage m we will have $j = i(\Gamma_m)$. But then either $\psi \in \Gamma_{m+1}$ or $\chi \in \Gamma_{m+1}$, contrary to the assumption that $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$.

Now suppose $\Gamma^* \vdash \varphi$. Then $\Gamma^* \vdash \varphi \vee \varphi$. But we've just proved that if $\Gamma^* \vdash \varphi \vee \varphi$ then $\varphi \in \Gamma^*$. Hence, Γ^* satisfies (1) of [Definition sc.1](#). \square

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Bibliography