sc.1 Lindenbaum's Lemma

int:sc:lin:sec The completeness theorem for intuitionistic logic is proved by assuming $\Gamma \nvdash \varphi$ and constructing a model $\mathfrak{M} \Vdash \Gamma$ and $\mathfrak{M} \nVdash \varphi$.

In classical logic the relation of derivability can be reduced to the notion of consistency since a formula φ is derivable from a set of formulas iff the set together with the negation of φ is inconsistent. This is not possible in intuitionistic logic. In intuitionistic logic, if $\neg\varphi$ is inconsistent, we only get that $\vdash \neg\neg\varphi$. Since $\neg\neg\varphi \to \varphi$ does not hold intuitionistically in general, we cannot conclude that $\vdash \varphi$.

Thus, when constructing the model $\mathfrak{M}$, we will need to keep track of the non-derivability of the formula φ and thus we will not be able to use a complete set $\Gamma^* \supseteq \Gamma$ to build the model $\mathfrak{M}$, as in every complete set Γ^*, we have $\Gamma^* \vdash \varphi \lor \neg\varphi$.

Instead of using a complete set Γ^*, we will us the notion of a prime set of formulas:

int:sc:lin:defn:prime **Definition sc.1.** A set of formulas Γ is *prime* iff

1. int:sc:lin:defn:prime1 Γ is consistent, i.e., $\Gamma \nvdash \bot$;
2. int:sc:lin:defn:prime2 if $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$; and
3. int:sc:lin:defn:prime3 if $\varphi \lor \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

int:sc:lin:lem:lindenbaum **Lemma sc.2 (Lindenbaum's Lemma).** *If $\Gamma \nvdash \varphi$, there is a $\Gamma^* \supseteq \Gamma$ such that Γ^* is prime and $\Gamma^* \nvdash \varphi$.*

Proof. Let $\psi_1 \lor \chi_1$, $\psi_2 \lor \chi_2$, …, be an enumeration of all formulas of the form $\psi \lor \chi$. We'll define an increasing sequence of sets of formulas Γ_n, where each Γ_{n+1} is defined as Γ_n together with one new formula. Γ^* will be the union of all Γ_n. The new formulas are selected so as to ensure that Γ^* is prime and still $\Gamma^* \nvdash \varphi$. This means that at each step we should find the first disjunction $\psi_i \lor \chi_i$ such that:

1. int:sc:lin:gamma-1 $\Gamma_n \vdash \psi_i \lor \chi_i$
2. int:sc:lin:gamma-2 $\psi_i \notin \Gamma_n$ and $\chi_i \notin \Gamma_n$

We add to Γ_n either ψ_i if $\Gamma_n \cup \{\psi_i\} \nvdash \varphi$, or χ_i otherwise. We'll have to show that this works. For now, let's define $i(n)$ as the least i such that (1) and (2) hold.

Define $\Gamma_0 = \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_{i(n)}\} & \text{if } \Gamma_n \cup \{\psi_{i(n)}\} \nvdash \varphi \\ \Gamma_n \cup \{\chi_{i(n)}\} & \text{otherwise} \end{cases}$$

If $i(n)$ is undefined, i.e., whenever $\Gamma_n \vdash \psi \lor \chi$, either $\psi \in \Gamma_n$ or $\chi \in \Gamma_n$, we let $\Gamma_{n+1} = \Gamma_n$. Now let $\Gamma^* = \bigcup_{n=0}^{\infty} \Gamma_n$

First we show that for all n, $\Gamma_n \nvdash \varphi$. We proceed by induction on n. For $n = 0$ the claim holds by the hypothesis of the theorem, i.e., $\Gamma \nvdash \varphi$. If $n > 0$, we have to show that if $\Gamma_n \nvdash \varphi$ then $\Gamma_{n+1} \nvdash \varphi$. If $i(n)$ is undefined, $\Gamma_{n+1} = \Gamma_n$ and there is nothing to prove. So suppose $i(n)$ is defined. For simplicity, let $i = i(n)$.

We'll prove the contrapositive of the claim. Suppose $\Gamma_{n+1} \vdash \varphi$. By construction, $\Gamma_{n+1} = \Gamma_n \cup \{\psi_i\}$ if $\Gamma_n \cup \{\psi_i\} \nvdash \varphi$, or else $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. It clearly can't be the first, since then $\Gamma_{n+1} \nvdash \varphi$. Hence, $\Gamma_n \cup \{\psi_i\} \vdash \varphi$ and $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. By definition of $i(n)$, we have that $\Gamma_n \vdash \psi_i \lor \chi_i$. We have $\Gamma_n \cup \{\psi_i\} \vdash \varphi$. We also have $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\} \vdash \varphi$. Hence, $\Gamma_n \vdash \varphi$, which is what we wanted to show.

If $\Gamma^* \vdash \varphi$, there would be some finite subset $\Gamma' \subseteq \Gamma^*$ such that $\Gamma' \vdash \varphi$. Each $\theta \in \Gamma'$ must be in Γ_i for some i. Let n be the largest of these. Since $\Gamma_i \subseteq \Gamma_n$ if $i \leq n$, $\Gamma' \subseteq \Gamma_n$. But then $\Gamma_n \vdash \varphi$, contrary to our proof above that $\Gamma_n \nvdash \varphi$.

Lastly, we show that Γ^* is prime, i.e., satisfies conditions (1), (2), and (3) of Definition sc.1.

First, $\Gamma^* \nvdash \varphi$, so Γ^* is consistent, so (1) holds.

We now show that if $\Gamma^* \vdash \psi \lor \chi$, then either $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$. This proves (3), since if $\psi \lor \chi \in \Gamma^*$ then also $\Gamma^* \vdash \psi \lor \chi$. So assume $\Gamma^* \vdash \psi \lor \chi$ but $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$. Since $\Gamma^* \vdash \psi \lor \chi$, $\Gamma_n \vdash \psi \lor \chi$ for some n. $\psi \lor \chi$ appears on the enumeration of all disjunctions, say, as $\psi_j \lor \chi_j$. $\psi_j \lor \chi_j$ satisfies the properties in the definition of $i(n)$, namely we have $\Gamma_n \vdash \psi_j \lor \chi_j$, while $\psi_j \notin \Gamma_n$ and $\chi_j \notin \Gamma_n$. At each stage, at least one fewer disjunction $\psi_i \lor \chi_i$ satisfies the conditions (since at each stage we add either ψ_i or χ_i), so at some stage m we will have $j = i(m)$. But then either $\psi \in \Gamma_{m+1}$ or $\chi \in \Gamma_{m+1}$, contrary to the assumption that $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$.

Now suppose $\Gamma^* \vdash \psi$. Then $\Gamma^* \vdash \psi \lor \psi$. But we've just proved that if $\Gamma^* \vdash \psi \lor \psi$ then $\psi \in \Gamma^*$. Hence, Γ^* satisfies (2) of Definition sc.1. $\square$

Problem sc.1. Show that if $\Gamma \nvdash \bot$ then Γ is consistent in classical logic, i.e., there is a valuation making all formulas in Γ true.

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Bibliography