Decidability

Observe that the proof of the completeness theorem gives us for every $\Gamma \not\vdash \varphi$ a model with an infinite number of worlds witnessing the fact that $\Gamma \not\models \varphi$. The following proposition shows that to prove $\models \varphi$ it is enough to prove that $\mathcal{M} \models \varphi$ for all finite models (i.e., models with a finite set of worlds).

**Theorem sc.1.** If $\not\models \varphi$ then there is a finite model $\mathcal{M}' \not\models \varphi$.

**Proof.** Assume $\mathcal{M} = \langle W, R, V \rangle$ is such that $\mathcal{M} \not\models \varphi$ and $P$ is the set of propositional variables occurring in $\varphi$. Define $\mathcal{M}' = \langle W', R', V' \rangle$ by letting $W' = \{[w] : w \in W\}$ where $[w] = \{p \in P : w \in V(p)\}$, $R'$ be the subset relation, and $V'(p) = \{[w] : p \in [w]\}$. It should be clear that $W'$ is a finite set and that $\mathcal{M}'$ is a relational model.

It can be shown, by induction on $\varphi$, that

$$\mathcal{M}, w \models \varphi \iff \mathcal{M}', [w] \models \varphi$$

for all formulas $\varphi$ with only propositional variables from $P$. This is left as an exercise for the reader.

**Problem sc.1.** Finish the proof of Theorem sc.1 by showing that $\mathcal{M}, w \models \varphi \iff \mathcal{M}', [w] \models \varphi$ for all formulas $\varphi$ with only propositional variables from $P$.

From Theorem sc.1 it follows that there is an algorithm to decide whether $\models \varphi$.

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Bibliography