sc.1 Decidability

int:sc:dec:sec Observe that the proof of the completeness theorem gives us for every $\Gamma \nvdash \varphi$ a model with an infinite number of worlds witnessing the fact that $\Gamma \nvDash \varphi$. The following proposition shows that to prove $\vDash \varphi$ it is enough to prove that $\mathfrak{M} \Vdash \varphi$ for all finite models (i.e., models with a finite set of worlds).

int:sc:dec:thm:decidability **Theorem sc.1.** *If $\nvDash \varphi$ then there is a finite model $\mathfrak{M}' \nVdash \varphi$.*

Proof. Assume $\mathfrak{M} = \langle W, R, V \rangle$ is such that $\mathfrak{M} \nVdash \varphi$ and P is the set of propositional variables occurring in φ. Define $\mathfrak{M}' = \langle W', R', V' \rangle$ by letting $W' = \{[w] : w \in W\}$ where $[w] = \{p \in P : w \in V(p)\}$, R' be the subset relation, and $V'(p) = \{[w] : p \in [w]\}$. It should be clear that W' is a finite set and that $\mathfrak{M}'$ is a relational model.

It can be shown, by induction on φ, that

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } \mathfrak{M}', [w] \Vdash \varphi$$

for all formulas φ with only propositional variables from P. This is left as an exercise for the reader. □

Problem sc.1. Finish the proof of Theorem sc.1 by showing that $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}', [w] \Vdash \varphi$ for all formulas φ with only propositional variables from P.

From Theorem sc.1 it follows that there is an algorithm to decide whether $\vDash \varphi$.

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Bibliography