sc.1 The Completeness Theorem

**Theorem sc.1.** If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

**Proof.** We prove the contrapositive: Suppose $\Gamma \not\models \varphi$. Then by ??, there is a prime set $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \not\models \varphi$. Consider the canonical model $\mathcal{M}(\Gamma^*)$ for $\Gamma^*$ as defined in ??, for any $\psi \in \Gamma$, $\Gamma^* \vdash \psi$. Note that $\Gamma^*(A) = \Gamma^*$. By the Truth Lemma (??), we have $\mathcal{M}(\Gamma^*)$, $A \models \psi$ for all $\psi \in \Gamma$ and $\mathcal{M}(\Gamma^*)$, $A \not\models \varphi$. This shows that $\Gamma \not\models \varphi$. \(\square\)

**Problem sc.1.** Show that if $\varphi$ only contains propositional variables, $\lor$, and $\land$, then $\not\models \varphi$. Use this to conclude that $\rightarrow$ is not definable in intuitionistic logic from $\lor$ and $\land$.

**Problem sc.2.** By using the completeness theorem prove that if $\vdash \varphi \lor \psi$ then $\vdash \varphi$ or $\vdash \psi$. (Hint: Assume $\mathcal{M}_1 \not\models \varphi$ and $\mathcal{M}_2 \not\models \psi$ and construct a new model $\mathcal{M}$ such that $\mathcal{M} \not\models \varphi \lor \psi$.)

**Problem sc.3.** Show that if $\mathcal{M}$ is a relational model using a linear order then $\mathcal{M} \models (\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)$.

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Bibliography