Another way to provide a semantics for intuitionistic logic is using the mathematical concept of a topology.

Definition sem.1. Let $X$ be a set. A topology on $X$ is a set $\mathcal{O} \subseteq \wp(X)$ that satisfies the properties below. The elements of $\mathcal{O}$ are called the open sets of the topology. The set $X$ together with $\mathcal{O}$ is called a topological space.

1. The empty set and the entire space open: $\emptyset, X \in \mathcal{O}$.
2. Open sets are closed under finite intersections: if $U, V \in \mathcal{O}$ then $U \cap V \in \mathcal{O}$.
3. Open sets are closed under arbitrary unions: if $U_i \in \mathcal{O}$ for all $i \in I$, then $\bigcup\{U_i : i \in I\} \in \mathcal{O}$.

We may write $X$ for a topology if the collection of open sets can be inferred from the context; note that, still, only after $X$ is endowed with open sets can it be called a topology.

Definition sem.2. A topological model of intuitionistic propositional logic is a triple $\mathfrak{X} = \langle X, \mathcal{O}, V \rangle$ where $\mathcal{O}$ is a topology on $X$ and $V$ is a function assigning an open set in $\mathcal{O}$ to each propositional variable.

Given a topological model $\mathfrak{X}$, we can define $[\varphi]_\mathfrak{X}$ inductively as follows:

1. $V(\bot) = \emptyset$
2. $[p]_\mathfrak{X} = V(p)$
3. $[\varphi \land \psi]_\mathfrak{X} = [\varphi]_\mathfrak{X} \cap [\psi]_\mathfrak{X}$
4. $[\varphi \lor \psi]_\mathfrak{X} = [\varphi]_\mathfrak{X} \cup [\psi]_\mathfrak{X}$
5. $[\varphi \rightarrow \psi]_\mathfrak{X} = \text{Int}((X \setminus [\varphi]_\mathfrak{X}) \cup [\psi]_\mathfrak{X})$

Here, $\text{Int}(V)$ is the function that maps a set $V \subseteq X$ to its interior; that is, the union of all open sets it contains. In other words,

$$\text{Int}(V) = \bigcup\{U : U \subseteq V \text{ and } U \in \mathcal{O}\}.$$

Note that the interior of any set is always open, since it is a union of open sets. Thus, $[\varphi]_\mathfrak{X}$ is always an open set.

Although topological semantics is highly abstract, there are ways to think about it that might motivate it. Suppose that the elements, or “points,” of $X$ are points at which statements can be evaluated. The set of all points where $\varphi$ is true is the proposition expressed by $\varphi$. Not every set of points is a potential proposition; only the elements of $\mathcal{O}$ are. $\varphi \vDash \psi$ if $\psi$ is true at every point at which $\varphi$ is true, i.e., $[\varphi]_\mathfrak{X} \subseteq [\psi]_\mathfrak{X}$, for all $X$. The absurd statement $\bot$ is never true, so $[\bot]_\mathfrak{X} = \emptyset$. How must the propositions expressed by $\psi \land \chi$, $\psi \lor \chi$, and $\psi \rightarrow \chi$ be related?
ψ → χ be related to those expressed by ψ and χ for the intuitionistically valid laws to hold, i.e., so that φ ⊨ ψ iff [φ]X ⊆ [ψ]X, ⊥ ⊨ φ for any φ, and only 0 ⊆ U for all U. Since ψ ∧ χ ⊨ ψ, [ψ ∧ χ]X ⊆ [ψ]X, and similarly [ψ ∧ χ]X ⊆ [χ]X.

The largest set satisfying W ⊆ U and W ⊆ V is U ∩ V. Conversely, ψ ⊨ ψ ∨ χ and χ ⊨ ψ ∨ χ, and so [ψ]X ⊆ [ψ ∨ χ]X and [χ]X ⊆ [ψ ∨ χ]X. The smallest set W such that U ⊆ W and V ⊆ W is U ∪ V. The definition for → is tricky: φ → ψ expresses the weakest proposition that, combined with φ, entails ψ. That φ → ψ combined with φ entails ψ is clear from (φ → ψ) ∧ φ ⊨ ψ. So [φ → ψ]X should be the greatest open set such that [φ → ψ]X ∩ [φ]X ⊆ [ψ]X, leading to our definition.

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Bibliography