Another way to provide a semantics for intuitionistic logic is using the mathematical concept of a topology.

**Definition sem.1.** Let $X$ be a set. A *topology on $X$* is a set $\mathcal{O} \subseteq \wp(X)$ that satisfies the properties below. The elements of $\mathcal{O}$ are called the *open sets* of the topology. The set $X$ together with $\mathcal{O}$ is called a *topological space*.

1. The empty set and the entire space open: $\emptyset, X \in \mathcal{O}$.
2. Open sets are closed under finite intersections: if $U, V \in \mathcal{O}$ then $U \cap V \in \mathcal{O}$.
3. Open sets are closed under arbitrary unions: if $U_i \in \mathcal{O}$ for all $i \in I$, then $\bigcup \{U_i : i \in I\} \in \mathcal{O}$.

We may write $X$ for a topology if the collection of open sets can be inferred from the context; note that, still, only after $X$ is endowed with open sets can it be called a topology.

**Definition sem.2.** A *topological model* of intuitionistic propositional logic is a triple $\mathfrak{X} = \langle X, \mathcal{O}, V \rangle$ where $\mathcal{O}$ is a topology on $X$ and $V$ is a function assigning an open set in $\mathcal{O}$ to each propositional variable.

Given a topological model $\mathfrak{X}$, we can define $[\varphi]_{\mathfrak{X}}$ inductively as follows:

1. $[\bot]_{\mathfrak{X}} = \emptyset$
2. $[p]_{\mathfrak{X}} = V(p)$
3. $[\varphi \land \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cap [\psi]_{\mathfrak{X}}$
4. $[\varphi \lor \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cup [\psi]_{\mathfrak{X}}$
5. $[\varphi \rightarrow \psi]_{\mathfrak{X}} = \text{Int}((X \setminus [\varphi]_{\mathfrak{X}}) \cup [\psi]_{\mathfrak{X}})$

Here, $\text{Int}(V)$ is the function that maps a set $V \subseteq X$ to its *interior*, that is, the union of all open sets it contains. In other words,

\[
\text{Int}(V) = \bigcup \{U : U \subseteq V \text{ and } U \in \mathcal{O}\}.
\]

Note that the interior of any set is always open, since it is a union of open sets. Thus, $[\varphi]_{\mathfrak{X}}$ is always an open set.

Although topological semantics is highly abstract, there are ways to think about it that might motivate it. Suppose that the elements, or “points,” of $X$ are points at which statements can be evaluated. The set of all points where $\varphi$ is true is the proposition expressed by $\varphi$. Not every set of points is a potential proposition; only the elements of $\mathcal{O}$ are. $\varphi \vdash \psi$ iff $\psi$ is true at every point at which $\varphi$ is true, i.e., $[\varphi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$, for all $X$. The absurd statement $\bot$ is never true, so $[\bot]_{\mathfrak{X}} = \emptyset$. How must the propositions expressed by $\psi \land \chi, \psi \lor \chi,$ and...
ψ → χ be related to those expressed by ψ and χ for the intuitionistically valid laws to hold, i.e., so that φ ⊨ ψ iff $[φ]_X \subseteq [ψ]_X$. ⊥ ⊨ φ for any φ, and only 0 ⊆ U for all U. Since ψ ∧ χ ⊨ ψ, $[ψ \land χ]_X \subseteq [ψ]_X$, and similarly $[ψ \land χ]_X \subseteq [χ]_X$.

The largest set satisfying $W \subseteq U$ and $W \subseteq V$ is $U \cap V$. Conversely, $ψ \vdash ψ \lor χ$ and $χ \vdash ψ \lor χ$, and so $[ψ]_X \subseteq [ψ \lor χ]_X$ and $[χ]_X \subseteq [ψ \lor χ]_X$. The smallest set $W$ such that $U \subseteq W$ and $V \subseteq W$ is $U \cup V$. The definition for → is tricky: φ → ψ expresses the weakest proposition that, combined with φ, entails ψ. That φ → ψ combined with φ entails ψ is clear from $(φ \rightarrow ψ) \land φ \vdash ψ$. So $[φ \rightarrow ψ]_X$ should be the greatest open set such that $[φ \rightarrow ψ]_X \cap [φ]_X \subset [ψ]_X$, leading to our definition.

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Bibliography