

## sem.1 Topological Semantics

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Another way to provide a semantics for intuitionistic logic is using the mathematical concept of a topology.

**Definition sem.1.** Let  $X$  be a set. A *topology on  $X$*  is a set  $\mathcal{O} \subseteq \wp(X)$  that satisfies the properties below. The **elements** of  $\mathcal{O}$  are called the *open sets* of the topology. The set  $X$  together with  $\mathcal{O}$  is called a *topological space*.

1. The empty set and the entire space open:  $\emptyset, X \in \mathcal{O}$ .
2. Open sets are closed under finite intersections: if  $U, V \in \mathcal{O}$  then  $U \cap V \in \mathcal{O}$
3. Open sets are closed under arbitrary unions: if  $U_i \in \mathcal{O}$  for all  $i \in I$ , then  $\bigcup\{U_i : i \in I\} \in \mathcal{O}$ .

We may write  $X$  for a topology if the collection of open sets can be inferred from the context; note that, still, only after  $X$  is endowed with open sets can it be called a topology.

**Definition sem.2.** A *topological model* of intuitionistic propositional logic is a triple  $\mathfrak{X} = \langle X, \mathcal{O}, V \rangle$  where  $\mathcal{O}$  is a topology on  $X$  and  $V$  is a function assigning an open set in  $\mathcal{O}$  to each propositional variable.

Given a topological model  $\mathfrak{X}$ , we can define  $[\varphi]_{\mathfrak{X}}$  inductively as follows:

1.  $V(\perp) = \emptyset$
2.  $[p]_{\mathfrak{X}} = V(p)$
3.  $[\varphi \wedge \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cap [\psi]_{\mathfrak{X}}$
4.  $[\varphi \vee \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cup [\psi]_{\mathfrak{X}}$
5.  $[\varphi \rightarrow \psi]_{\mathfrak{X}} = \text{Int}((X \setminus [\varphi]_{\mathfrak{X}}) \cup [\psi]_{\mathfrak{X}})$

Here,  $\text{Int}(V)$  is the function that maps a set  $V \subseteq X$  to its *interior*, that is, the union of all open sets it contains. In other words,

$$\text{Int}(V) = \bigcup\{U : U \subseteq V \text{ and } U \in \mathcal{O}\}.$$

Note that the interior of any set is always open, since it is a union of open sets. Thus,  $[\varphi]_{\mathfrak{X}}$  is always an open set.

Although topological semantics is highly abstract, there are ways to think about it that might motivate it. Suppose that the **elements**, or “points,” of  $X$  are points at which statements can be evaluated. The set of all points where  $\varphi$  is true is the proposition expressed by  $\varphi$ . Not every set of points is a potential proposition; only the **elements** of  $\mathcal{O}$  are.  $\varphi \vDash \psi$  iff  $\psi$  is true at every point at which  $\varphi$  is true, i.e.,  $[\varphi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$ , for all  $X$ . The absurd statement  $\perp$  is never true, so  $[\perp]_{\mathfrak{X}} = \emptyset$ . How must the propositions expressed by  $\psi \wedge \chi$ ,  $\psi \vee \chi$ , and

$\psi \rightarrow \chi$  be related to those expressed by  $\psi$  and  $\chi$  for the intuitionistically valid laws to hold, i.e., so that  $\varphi \vdash \psi$  iff  $[\varphi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$ .  $\perp \vdash \varphi$  for any  $\varphi$ , and only  $\emptyset \subseteq U$  for all  $U$ . Since  $\psi \wedge \chi \vdash \psi$ ,  $[\psi \wedge \chi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$ , and similarly  $[\psi \wedge \chi]_{\mathfrak{X}} \subseteq [\chi]_{\mathfrak{X}}$ . The largest set satisfying  $W \subseteq U$  and  $W \subseteq V$  is  $U \cap V$ . Conversely,  $\psi \vdash \psi \vee \chi$  and  $\chi \vdash \psi \vee \chi$ , and so  $[\psi]_{\mathfrak{X}} \subseteq [\psi \vee \chi]_{\mathfrak{X}}$  and  $[\chi]_{\mathfrak{X}} \subseteq [\psi \vee \chi]_{\mathfrak{X}}$ . The smallest set  $W$  such that  $U \subseteq W$  and  $V \subseteq W$  is  $U \cup V$ . The definition for  $\rightarrow$  is tricky:  $\varphi \rightarrow \psi$  expresses the weakest proposition that, combined with  $\varphi$ , entails  $\psi$ . That  $\varphi \rightarrow \psi$  combined with  $\varphi$  entails  $\psi$  is clear from  $(\varphi \rightarrow \psi) \wedge \varphi \vdash \psi$ . So  $[\varphi \rightarrow \psi]_{\mathfrak{X}}$  should be the greatest open set such that  $[\varphi \rightarrow \psi]_{\mathfrak{X}} \cap [\varphi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$ , leading to our definition.

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## Bibliography