sem.1 Topological Semantics

int:sem:top:sec Another way to provide a semantics for intuitionistic logic is using the mathematical concept of a topology.

Definition sem.1. Let X be a set. A *topology on* X is a set $\mathcal{O} \subseteq \wp(X)$ that satisfies the properties below. The elements of $\mathcal{O}$ are called the *open sets* of the topology. The set X together with $\mathcal{O}$ is called a *topological space*.

1. The empty set and the entire space are open: $\emptyset$, $X \in \mathcal{O}$.
2. Open sets are closed under finite intersections: if U, $V \in \mathcal{O}$ then $U \cap V \in \mathcal{O}$
3. Open sets are closed under arbitrary unions: if $U_i \in \mathcal{O}$ for all $i \in I$, then $\bigcup\{U_i : i \in I\} \in \mathcal{O}$.

We may write X for a topology if the collection of open sets can be inferred from the context; note that, still, only after X is endowed with open sets can it be called a topology.

Definition sem.2. A *topological model* of intuitionistic propositional logic is a triple $\mathfrak{X} = \langle X, \mathcal{O}, V\rangle$ where $\mathcal{O}$ is a topology on X and V is a function assigning an open set in $\mathcal{O}$ to each propositional variable.

Given a topological model $\mathfrak{X}$, we can define $[\varphi]_{\mathfrak{X}}$ inductively as follows:

1. $[\bot]_{\mathfrak{X}} = \emptyset$
2. $[p]_{\mathfrak{X}} = V(p)$
3. $[\varphi \wedge \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cap [\psi]_{\mathfrak{X}}$
4. $[\varphi \vee \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cup [\psi]_{\mathfrak{X}}$
5. $[\varphi \to \psi]_{\mathfrak{X}} = \mathrm{Int}((X \setminus [\varphi]_{\mathfrak{X}}) \cup [\psi]_{\mathfrak{X}})$

Here, $\mathrm{Int}(V)$ is the function that maps a set $V \subseteq X$ to its *interior*, that is, the union of all open sets it contains. In other words,

$$\mathrm{Int}(V) = \bigcup\{U : U \subseteq V \text{ and } U \in \mathcal{O}\}.$$

Note that the interior of any set is always open, since it is a union of open sets. Thus, $[\varphi]_{\mathfrak{X}}$ is always an open set.

Although topological semantics is highly abstract, there are ways to think about it that might motivate it. Suppose that the elements, or "points," of X are points at which statements can be evaluated. The set of all points where φ is true is the proposition expressed by φ. Not every set of points is a potential proposition; only the elements of $\mathcal{O}$ are. $\varphi \vDash \psi$ iff ψ is true at every point at which φ is true, i.e., $[\varphi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$, for all X. The absurd statement $\bot$ is never true, so $[\bot]_{\mathfrak{X}} = \emptyset$.

How must the propositions expressed by $\psi \wedge \chi$, $\psi \vee \chi$, and $\psi \to \chi$ be related to those expressed by ψ and χ for the intuitionistically valid laws to hold, i.e., so that $\varphi \vdash \psi$ iff $[\varphi]_{\mathfrak{X}} \subset [\psi]_{\mathfrak{X}}$? We require $\bot \vdash \varphi$ for any φ, which is satisfied because $\emptyset \subseteq U$ for all U. Since $\psi \wedge \chi \vdash \psi$, we require that $[\psi \wedge \chi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$, and similarly $[\psi \wedge \chi]_{\mathfrak{X}} \subseteq [\chi]_{\mathfrak{X}}$. The largest set satisfying $W \subseteq U$ and $W \subseteq V$ is $U \cap V$. Conversely, $\psi \vdash \psi \vee \chi$ and $\chi \vdash \psi \vee \chi$, and so we require that $[\psi]_{\mathfrak{X}} \subseteq [\psi \vee \chi]_{\mathfrak{X}}$ and $[\chi]_{\mathfrak{X}} \subseteq [\psi \vee \chi]_{\mathfrak{X}}$. The smallest set W such that $U \subseteq W$ and $V \subseteq W$ is $U \cup V$.

The definition for $\to$ is tricky: $\varphi \to \psi$ expresses the weakest proposition that, combined with φ, entails ψ. That $\varphi \to \psi$ combined with φ entails ψ is clear from $(\varphi \to \psi) \wedge \varphi \vdash \psi$. So $[\varphi \to \psi]_{\mathfrak{X}}$ should be the greatest open set such that $[\varphi \to \psi]_{\mathfrak{X}} \cap [\varphi]_{\mathfrak{X}} \subset [\psi]_{\mathfrak{X}}$, leading to our definition.

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Bibliography