

# Chapter udf

## Semantics

This chapter collects definitions for semantics for intuitionistic logic. So far only Kripke and topological semantics are covered. There are no examples yet, either of how models make formulas true or of proofs that formulas are valid.

### sem.1 Introduction

int:sem:int:  
sec No logic is satisfactorily described without a semantics, and intuitionistic logic is no exception. Whereas for classical logic, the semantics based on **valuations** is canonical, there are several competing semantics for intuitionistic logic. None of them are completely satisfactory in the sense that they give an intuitionistically acceptable account of the meanings of the connectives.

The semantics based on **relational models**, similar to the semantics for modal logics, is perhaps the most popular one. In this semantics, **propositional variables** are assigned to worlds, and these worlds are related by an accessibility relation. That relation is always a partial order, i.e., it is reflexive, antisymmetric, and transitive.

Intuitively, you might think of these worlds as states of knowledge or “evidentiary situations.” A state  $w'$  is accessible from  $w$  iff, for all we know,  $w'$  is a possible (future) state of knowledge, i.e., one that is compatible with what’s known at  $w$ . Once a proposition is known, it can’t become un-known, i.e., whenever  $\varphi$  is known at  $w$  and  $Rww'$ ,  $\varphi$  is known at  $w'$  as well. So “knowledge” is monotonic with respect to the accessibility relation.

If we define “ $\varphi$  is known” as in epistemic logic as “true in all epistemic alternatives,” then  $\varphi \wedge \psi$  is known at  $w$  if in all epistemic alternatives, both  $\varphi$  and  $\psi$  are known. But since knowledge is monotonic and  $R$  is reflexive, that means that  $\varphi \wedge \psi$  is known at  $w$  iff  $\varphi$  and  $\psi$  are known at  $w$ . For the same reason,  $\varphi \vee \psi$  is known at  $w$  iff at least one of them is known. So for  $\wedge$  and  $\vee$ , the truth conditions of the connectives coincide with those in classical logic.

The truth conditions for the conditional, however, differ from classical logic.  $\varphi \rightarrow \psi$  is known at  $w$  iff at no  $w'$  with  $Rww'$ ,  $\varphi$  is known without  $\psi$  also being known. This is not the same as the condition that  $\varphi$  is unknown or  $\psi$  is known at  $w$ . For if we know neither  $\varphi$  nor  $\psi$  at  $w$ , there might be a future epistemic state  $w'$  with  $Rww'$  such that at  $w'$ ,  $\varphi$  is known without also coming to know  $\psi$ .

We know  $\neg\varphi$  only if there is no possible future epistemic state in which we know  $\varphi$ . Here the idea is that if  $\varphi$  were knowable, then in some possible future epistemic state  $\varphi$  becomes known. Since we can't know  $\perp$ , in that future epistemic state, we would know  $\varphi$  but not know  $\perp$ .

On this interpretation the principle of excluded middle fails. For there are some  $\varphi$  which we don't yet know, but which we might come to know. For such a formula  $\varphi$ , both  $\varphi$  and  $\neg\varphi$  are unknown, so  $\varphi \vee \neg\varphi$  is not known. But we do know, e.g., that  $\neg(\varphi \wedge \neg\varphi)$ . For no future state in which we know both  $\varphi$  and  $\neg\varphi$  is possible, and we know this independently of whether or not we know  $\varphi$  or  $\neg\varphi$ .

**Relational models** are not the only available semantics for intuitionistic logic. The topological semantics is another: here propositions are interpreted as open sets in a topological space, and the connectives are interpreted as operations on these sets (e.g.,  $\wedge$  corresponds to intersection).

## sem.2 Relational models

In order to give a precise semantics for intuitionistic propositional logic, we have to give a definition of what counts as a model relative to which we can evaluate formulas. On the basis of such a definition it is then also possible to define semantics notions such as validity and entailment. One such semantics is given by **relational models**.

int:sem:rel:  
sec

**Definition sem.1.** A **relational model** for intuitionistic propositional logic is a triple  $\mathfrak{M} = \langle W, R, V \rangle$ , where

1.  $W$  is a non-empty set,
2.  $R$  is a partial order (i.e., a reflexive, antisymmetric, and transitive binary relation) on  $W$ , and
3.  $V$  is a function assigning to each **propositional variable**  $p$  a subset of  $W$ , such that
4.  $V$  is monotone with respect to  $R$ , i.e., if  $w \in V(p)$  and  $Rww'$ , then  $w' \in V(p)$ .

**Definition sem.2.** We define the notion of  $\varphi$  *being true at  $w$  in  $\mathfrak{M}$* ,  $\mathfrak{M}, w \Vdash \varphi$ , inductively as follows:

int:sem:rel:  
defn:true-at-w

1.  $\varphi \equiv p$ :  $\mathfrak{M}, w \Vdash \varphi$  iff  $w \in V(p)$ .
2.  $\varphi \equiv \perp$ : not  $\mathfrak{M}, w \Vdash \varphi$ .

3.  $\varphi \equiv \neg\psi$ :  $\mathfrak{M}, w \Vdash \varphi$  iff for no  $w'$  such that  $Rww'$ ,  $\mathfrak{M}, w' \Vdash \psi$ .
4.  $\varphi \equiv \psi \wedge \chi$ :  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}, w \Vdash \psi$  and  $\mathfrak{M}, w \Vdash \chi$ .
5.  $\varphi \equiv \psi \vee \chi$ :  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}, w \Vdash \psi$  or  $\mathfrak{M}, w \Vdash \chi$  (or both).
6.  $\varphi \equiv \psi \rightarrow \chi$ :  $\mathfrak{M}, w \Vdash \varphi$  iff for every  $w'$  such that  $Rww'$ , not  $\mathfrak{M}, w' \Vdash \psi$  or  $\mathfrak{M}, w' \Vdash \chi$  (or both).

We write  $\mathfrak{M}, w \not\Vdash \varphi$  if not  $\mathfrak{M}, w \Vdash \varphi$ . If  $\Gamma$  is a set of **formulas**,  $\mathfrak{M}, w \Vdash \Gamma$  means  $\mathfrak{M}, w \Vdash \psi$  for all  $\psi \in \Gamma$ .

**Problem sem.1.** Show that according to **Definition sem.2**,  $\mathfrak{M}, w \Vdash \neg\varphi$  iff  $\mathfrak{M}, w \Vdash \varphi \rightarrow \perp$ .

*int:sem:rel:  
prop:true-monotonic*

**Proposition sem.3.** *Truth at worlds is monotonic with respect to  $R$ , i.e., if  $\mathfrak{M}, w \Vdash \varphi$  and  $Rww'$ , then  $\mathfrak{M}, w' \Vdash \varphi$ .*

*Proof.* Exercise. □

**Problem sem.2.** Prove **Proposition sem.3**.

### sem.3 Semantic Notions

*int:sem:sem:  
sec*

**Definition sem.4.** We say  $\varphi$  is *true in the model*  $\mathfrak{M} = \langle W, R, V \rangle$ ,  $\mathfrak{M} \Vdash \varphi$ , iff  $\mathfrak{M}, w \Vdash \varphi$  for all  $w \in W$ .  $\varphi$  is *valid*,  $\vDash \varphi$ , iff it is true in all models. We say a set of **formulas**  $\Gamma$  *entails*  $\varphi$ ,  $\Gamma \vDash \varphi$ , iff for every model  $\mathfrak{M}$  and every  $w$  such that  $\mathfrak{M}, w \Vdash \Gamma$ ,  $\mathfrak{M}, w \Vdash \varphi$ .

*int:sem:sem:  
prop:sat-entails  
int:sem:sem:  
prop:sat-entails1  
int:sem:sem:  
prop:sat-entails2*

**Proposition sem.5.**

1. If  $\mathfrak{M}, w \Vdash \Gamma$  and  $\Gamma \vDash \varphi$ , then  $\mathfrak{M}, w \Vdash \varphi$ .
2. If  $\mathfrak{M} \Vdash \Gamma$  and  $\Gamma \vDash \varphi$ , then  $\mathfrak{M} \Vdash \varphi$ .

*Proof.* 1. Suppose  $\mathfrak{M} \Vdash \Gamma$ . Since  $\Gamma \vDash \varphi$ , we know that if  $\mathfrak{M}, w \Vdash \Gamma$ , then  $\mathfrak{M}, w \Vdash \varphi$ . Since  $\mathfrak{M}, u \Vdash \Gamma$  for all every  $u \in W$ ,  $\mathfrak{M}, w \Vdash \Gamma$ . Hence  $\mathfrak{M}, w \Vdash \varphi$ .

2. Follows immediately from (1). □

*int:sem:sem:  
defn:restrict*

**Definition sem.6.** Suppose  $\mathfrak{M}$  is a relational model and  $w \in W$ . The *restriction*  $\mathfrak{M}_w = \langle W_w, R_w, V_w \rangle$  of  $\mathfrak{M}$  to  $w$  is given by:

$$\begin{aligned} W_w &= \{u \in W : Rwu\}, \\ R_w &= R \cap (W_w)^2, \text{ and} \\ V_w(p) &= V(p) \cap W_w. \end{aligned}$$

**Proposition sem.7.**  $\mathfrak{M}, w \Vdash \varphi$  iff  $\mathfrak{M}_w \Vdash \varphi$ .

*int.sem.sem:  
prop.restrict*

**Problem sem.3.** Prove [Proposition sem.7](#).

**Proposition sem.8.** Suppose for every model  $\mathfrak{M}$  such that  $\mathfrak{M} \Vdash \Gamma$ ,  $\mathfrak{M} \Vdash \varphi$ . Then  $\Gamma \vDash \varphi$ .

*Proof.* Suppose that  $\mathfrak{M}, w \Vdash \Gamma$ . By the [Proposition sem.7](#) applied to every  $\psi \in \Gamma$ , we have  $\mathfrak{M}_w \Vdash \Gamma$ . By the assumption, we have  $\mathfrak{M}_w \Vdash \varphi$ . By [Proposition sem.7](#) again, we get  $\mathfrak{M}, w \Vdash \varphi$ .  $\square$

## sem.4 Topological Semantics

Another way to provide a semantics for intuitionistic logic is using the mathematical concept of a topology. *int.sem:top:sec*

**Definition sem.9.** Let  $X$  be a set. A *topology on  $X$*  is a set  $\mathcal{O} \subseteq \wp(X)$  that satisfies the properties below. The *elements* of  $\mathcal{O}$  are called the *open sets* of the topology. The set  $X$  together with  $\mathcal{O}$  is called a *topological space*.

1. The empty set and the entire space are open:  $\emptyset, X \in \mathcal{O}$ .
2. Open sets are closed under finite intersections: if  $U, V \in \mathcal{O}$  then  $U \cap V \in \mathcal{O}$ .
3. Open sets are closed under arbitrary unions: if  $U_i \in \mathcal{O}$  for all  $i \in I$ , then  $\bigcup\{U_i : i \in I\} \in \mathcal{O}$ .

We may write  $X$  for a topology if the collection of open sets can be inferred from the context; note that, still, only after  $X$  is endowed with open sets can it be called a topology.

**Definition sem.10.** A *topological model* of intuitionistic propositional logic is a triple  $\mathfrak{X} = \langle X, \mathcal{O}, V \rangle$  where  $\mathcal{O}$  is a topology on  $X$  and  $V$  is a function assigning an open set in  $\mathcal{O}$  to each propositional variable.

Given a topological model  $\mathfrak{X}$ , we can define  $[\varphi]_{\mathfrak{X}}$  inductively as follows:

1.  $[\perp]_{\mathfrak{X}} = \emptyset$
2.  $[p]_{\mathfrak{X}} = V(p)$
3.  $[\varphi \wedge \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cap [\psi]_{\mathfrak{X}}$
4.  $[\varphi \vee \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cup [\psi]_{\mathfrak{X}}$
5.  $[\varphi \rightarrow \psi]_{\mathfrak{X}} = \text{Int}((X \setminus [\varphi]_{\mathfrak{X}}) \cup [\psi]_{\mathfrak{X}})$

Here,  $\text{Int}(V)$  is the function that maps a set  $V \subseteq X$  to its *interior*, that is, the union of all open sets it contains. In other words,

$$\text{Int}(V) = \bigcup\{U : U \subseteq V \text{ and } U \in \mathcal{O}\}.$$

Note that the interior of any set is always open, since it is a union of open sets. Thus,  $[\varphi]_{\mathbf{x}}$  is always an open set.

Although topological semantics is highly abstract, there are ways to think about it that might motivate it. Suppose that the **elements**, or “points,” of  $X$  are points at which statements can be evaluated. The set of all points where  $\varphi$  is true is the proposition expressed by  $\varphi$ . Not every set of points is a potential proposition; only the **elements** of  $\mathcal{O}$  are.  $\varphi \models \psi$  iff  $\psi$  is true at every point at which  $\varphi$  is true, i.e.,  $[\varphi]_{\mathbf{x}} \subseteq [\psi]_{\mathbf{x}}$ , for all  $X$ . The absurd statement  $\perp$  is never true, so  $[\perp]_{\mathbf{x}} = \emptyset$ .

How must the propositions expressed by  $\psi \wedge \chi$ ,  $\psi \vee \chi$ , and  $\psi \rightarrow \chi$  be related to those expressed by  $\psi$  and  $\chi$  for the intuitionistically valid laws to hold, i.e., so that  $\varphi \vdash \psi$  iff  $[\varphi]_{\mathbf{x}} \subseteq [\psi]_{\mathbf{x}}$ ? We require  $\perp \vdash \varphi$  for any  $\varphi$ , which is satisfied because  $\emptyset \subseteq U$  for all  $U$ . Since  $\psi \wedge \chi \vdash \psi$ , we require that  $[\psi \wedge \chi]_{\mathbf{x}} \subseteq [\psi]_{\mathbf{x}}$ , and similarly  $[\psi \wedge \chi]_{\mathbf{x}} \subseteq [\chi]_{\mathbf{x}}$ . The largest set satisfying  $W \subseteq U$  and  $W \subseteq V$  is  $U \cap V$ . Conversely,  $\psi \vdash \psi \vee \chi$  and  $\chi \vdash \psi \vee \chi$ , and so we require that  $[\psi]_{\mathbf{x}} \subseteq [\psi \vee \chi]_{\mathbf{x}}$  and  $[\chi]_{\mathbf{x}} \subseteq [\psi \vee \chi]_{\mathbf{x}}$ . The smallest set  $W$  such that  $U \subseteq W$  and  $V \subseteq W$  is  $U \cup V$ .

The definition for  $\rightarrow$  is tricky:  $\varphi \rightarrow \psi$  expresses the weakest proposition that, combined with  $\varphi$ , entails  $\psi$ . That  $\varphi \rightarrow \psi$  combined with  $\varphi$  entails  $\psi$  is clear from  $(\varphi \rightarrow \psi) \wedge \varphi \vdash \psi$ . So  $[\varphi \rightarrow \psi]_{\mathbf{x}}$  should be the greatest open set such that  $[\varphi \rightarrow \psi]_{\mathbf{x}} \cap [\varphi]_{\mathbf{x}} \subseteq [\psi]_{\mathbf{x}}$ , leading to our definition.

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# Bibliography