Let us write \( \Gamma \Rightarrow \phi \) if there is a natural deduction derivation with \( \Gamma \) as undischarged assumptions and \( \phi \) as conclusion; or \( \Rightarrow \phi \) if \( \Gamma \) is empty.

We write \( \Gamma, \phi_1, \ldots, \phi_n \) for \( \Gamma \cup \{ \phi_1, \ldots, \phi_n \} \), and \( \Gamma, \Delta \) for \( \Gamma \cup \Delta \).

Observe that when we have \( \Gamma \Rightarrow \phi \land \phi \), meaning we have a derivation with \( \Gamma \) as undischarged assumptions and \( \phi \land \phi \) as end-formula, then by applying \( \land \)Elim at the bottom, we can get a derivation with the same undischarged assumptions and \( \phi \) as conclusion. In other words, if \( \Gamma \Rightarrow \phi \land \psi \), then \( \Gamma \Rightarrow \phi \).

The label \( \land \)Elim hints at the relation with the rule of the same name in natural deduction.

Likewise, suppose we have \( \Gamma, \phi \Rightarrow \psi \), meaning we have a derivation with undischarged assumptions \( \Gamma, \phi \) and end-formula \( \psi \). If we apply the \( \rightarrow \)Intro rule, we have a derivation with \( \Gamma \) as undischarged assumptions and \( \phi \rightarrow \psi \) as the end-formula, i.e., \( \Gamma \Rightarrow \phi \rightarrow \psi \). Note how this has made the discharge of assumptions more explicit.

We can draw conclusions from other rules in the same fashion, which is spelled out as follows:

\[
\frac{\Gamma \Rightarrow \phi, \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \phi \land \psi} \quad \land \text{Intro}
\]

\[
\frac{\Gamma \Rightarrow \phi \land \psi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi} \quad \land \text{Elim}_1
\]

\[
\frac{\Gamma \Rightarrow \phi \land \psi \quad \Gamma \Rightarrow \phi}{\Gamma \Rightarrow \psi} \quad \land \text{Elim}_2
\]

\[
\frac{\Gamma \Rightarrow \phi \land \psi}{\Gamma \Rightarrow \phi \lor \psi} \quad \lor \text{Intro}_1
\]

\[
\frac{\Delta \Rightarrow \phi \land \psi \quad \Gamma \Rightarrow \phi \land \psi}{\Gamma \Rightarrow \phi} \quad \lor \text{Elim}_2
\]

\[
\frac{\Gamma \Rightarrow \phi \lor \psi \quad \Delta, \phi \Rightarrow \chi \quad \Delta', \psi \Rightarrow \chi}{\Gamma, \Delta, \Delta' \Rightarrow \chi} \quad \lor \text{Elim}
\]

Any assumption by itself is a derivation of \( \phi \) from \( \phi \), i.e., we always have \( \phi \Rightarrow \phi \).

Together, these rules can be taken as a calculus about what natural deduction derivations exist. They can also be taken as a notational variant of natural deduction, in which each step records not only the formula derived but also the undischarged assumptions from which it was derived.
\[
\begin{align*}
\varphi \Rightarrow & \varphi \\
\varphi \Rightarrow & \varphi \lor (\varphi \rightarrow \bot) \quad \psi \Rightarrow \psi \\
\varphi, \psi \Rightarrow & \Rightarrow \bot \\
(\psi \Rightarrow & \varphi \rightarrow \bot) \\
(\psi \Rightarrow & \varphi \lor (\varphi \rightarrow \bot) \quad (\psi \Rightarrow \psi \\
(\psi \Rightarrow & \bot) \\
& \Rightarrow \psi \rightarrow \bot
\end{align*}
\]

where \(\psi\) is short for \((\varphi \lor (\varphi \rightarrow \bot)) \rightarrow \bot\).

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Bibliography