Let us write $\Gamma \Rightarrow \phi$ if there is a natural deduction derivation with $\Gamma$ as undischarged assumptions and $\phi$ as conclusion; or $\Rightarrow \phi$ if $\Gamma$ is empty.

We write $\Gamma, \phi_1, \ldots, \phi_n$ for $\Gamma \cup \{\phi_1, \ldots, \phi_n\}$, and $\Gamma, \Delta$ for $\Gamma \cup \Delta$.

Observe that when we have $\Gamma \Rightarrow \phi \land \psi$, meaning we have a derivation with $\Gamma$ as undischarged assumptions and $\phi \land \psi$ as end-formula, then by applying $\land$Elim at the bottom, we can get a derivation with the same undischarged assumptions and $\phi$ as conclusion. In other words, if $\Gamma \Rightarrow \phi \land \psi$, then $\Gamma \Rightarrow \phi$.

$$\frac{\Gamma \Rightarrow \phi \land \psi}{\Gamma \Rightarrow \phi} \land\text{Elim}$$

The label $\land$Elim hints at the relation with the rule of the same name in natural deduction.

Likewise, suppose we have $\Gamma, \varphi \Rightarrow \psi$, meaning we have a derivation with undischarged assumptions $\Gamma, \varphi$ and end-formula $\psi$. If we apply the $\Rightarrow$Intro rule, we have a derivation with $\Gamma$ as undischarged assumptions and $\varphi \Rightarrow \psi$ as the end-formula, i.e., $\Gamma \Rightarrow \varphi \Rightarrow \psi$. Note how this has made the discharge of assumptions more explicit.

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \Rightarrow \psi} \Rightarrow\text{Intro}$$

We can draw conclusions from other rules in the same fashion, which is spelled out as follows:

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma, \Delta \Rightarrow \varphi \land \psi} \land\text{Intro}$$

$$\frac{\Gamma \Rightarrow \phi \land \psi}{\Gamma \Rightarrow \phi} \land\text{Elim}_1$$

$$\frac{\Gamma \Rightarrow \phi \land \psi}{\Gamma \Rightarrow \psi} \land\text{Elim}_2$$

$$\frac{\Gamma \Rightarrow \varphi \lor \psi}{\Gamma \Rightarrow \varphi} \lor\text{Intro}_1$$

$$\frac{\Gamma \Rightarrow \varphi \lor \psi}{\Gamma \Rightarrow \psi} \lor\text{Intro}_2$$

$$\frac{\Gamma \Rightarrow \varphi \lor \psi}{\Delta, \varphi \Rightarrow \chi} \lor\text{Elim}$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \Rightarrow \psi} \Rightarrow\text{Intro}$$

$$\frac{\Delta \Rightarrow \varphi \Rightarrow \psi}{\Delta \Rightarrow \psi} \Rightarrow\text{Elim}$$

Any assumption by itself is a derivation of $\varphi$ from $\varphi$, i.e., we always have $\varphi \Rightarrow \varphi$.

Together, these rules can be taken as a calculus about what natural deduction derivations exist. They can also be taken as a notational variant of natural deduction, in which each step records not only the formula derived but also the undischarged assumptions from which it was derived.
\[
\frac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \lor (\varphi \rightarrow \bot)} \quad \psi \Rightarrow \psi
\]
\[
\varphi, \psi \Rightarrow \Rightarrow \bot
\]
\[
(\psi \Rightarrow \varphi \rightarrow \bot)
\]
\[
(\psi \Rightarrow \varphi \lor (\varphi \rightarrow \bot)) \quad (\psi \Rightarrow \psi)
\]
\[
(\psi \Rightarrow \bot)
\]
\[
\Rightarrow \psi \rightarrow \bot
\]

where $\psi$ is short for $(\varphi \lor (\varphi \rightarrow \bot)) \rightarrow \bot$.

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**Bibliography**