

## pty.1 Converting Derivations to Proof Terms

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sec

We will describe the process of converting natural deduction **derivations** to pairs. We will write a proof term to the left of each formula in the **derivation**, resulting in expressions of the form  $M : \varphi$ . We'll then say that,  $M$  *witnesses*  $\varphi$ . Let's call such an expression a *judgment*.

First let us assign to each assumption a variable, with the following constraints:

1. Assumptions **discharged** in the same step (that is, with the same number on the square bracket) must be assigned the same variable.
2. For assumptions not **discharged**, assumptions of different **formulas** should be assigned different variables.

Such an assignment translates all assumptions of the form

$$\varphi \quad \text{into} \quad x : \varphi.$$

With assumptions all associated with variables (which are terms), we can now inductively translate the rest of the deduction tree. The modified natural deduction rules taking into account context and proof terms are given below. Given the proof terms for the premise(s), we obtain the corresponding proof term for conclusion.

$$\frac{M_1 : \varphi_1 \quad M_2 : \varphi_2}{\langle M_1, M_2 \rangle : \varphi_1 \wedge \varphi_2} \wedge\text{Intro}$$

$$\frac{M : \varphi_1 \wedge \varphi_2}{p_i(M) : \varphi_1} \wedge\text{Elim}_1 \quad \frac{M : \varphi_1 \wedge \varphi_2}{p_i(M) : \varphi_2} \wedge\text{Elim}_2$$

In  $\wedge\text{Intro}$  we assume we have  $\varphi_1$  witnessed by term  $M_1$  and  $\varphi_2$  witnessed by term  $M_2$ . We pack up the two terms into a pair  $\langle M_1, M_2 \rangle$  which witnesses  $\varphi_1 \wedge \varphi_2$ .

In  $\wedge\text{Elim}_i$  we assume that  $M$  witnesses  $\varphi_1 \wedge \varphi_2$ . The term witnessing  $\varphi_i$  is  $p_i(M)$ . Note that  $M$  is not necessary of the form  $\langle M_1, M_2 \rangle$ , so we cannot simply assign  $M_1$  to the conclusion  $\varphi_i$ .

Note how this coincides with the BHK interpretation. What the BHK interpretation does not specify is how the function used as proof for  $\varphi \rightarrow \psi$  is supposed to be obtained. If we think of proof terms as proofs or functions of proofs, we can be more explicit.

$$\frac{\begin{array}{c} [x : \varphi] \\ \vdots \\ \vdots \\ N : \psi \end{array}}{\lambda x^\varphi. N : \varphi \rightarrow \psi} \rightarrow\text{Intro} \quad \frac{P : \varphi \rightarrow \psi \quad Q : \varphi}{PQ : \psi} \rightarrow\text{Elim}$$

The  $\lambda$  notation should be understood as the same as in the lambda calculus, and  $PQ$  means applying  $P$  to  $Q$ .

$$\frac{\frac{M_1 : \varphi_1}{\text{in}_1^{\varphi_1}(M_1) : \varphi_1 \vee \varphi_2} \vee\text{Intro}_1 \quad \frac{M_2 : \varphi_2}{\text{in}_2^{\varphi_2}(M_2) : \varphi_1 \vee \varphi_2} \vee\text{Intro}_2}{\frac{M : A_1 \vee \varphi_2 \quad N_1 : \chi \quad N_2 : \chi}{\text{case}(M, x_1.N_1, x_2.N_2) : \chi} \vee\text{Elim}}$$

$[x_1 : \varphi_1]$        $[x_2 : \varphi_2]$   
 $\vdots$                        $\vdots$   
 $\vdots$                        $\vdots$

The proof term  $\text{in}_1^{\varphi_1}(M_1)$  is a term witnessing  $\varphi_1 \vee \varphi_2$ , where  $M_1$  witnesses  $\varphi_1$ .

The term  $\text{case}(M, x_1.N_1, x_2.N_2)$  mimics the case clause in programming languages: we already have the **derivation** of  $\varphi \vee \psi$ , a **derivation** of  $\chi$  assuming  $\varphi$ , and a **derivation** of  $\chi$  assuming  $\psi$ . The *case* operator thus select the appropriate proof depending on  $M$ ; either way it's a proof of  $\chi$ .

$$\frac{N : \perp}{\text{contr}_\varphi(N) : \varphi} \perp_I$$

$\text{contr}_\varphi(N)$  is a term witnessing  $\varphi$ , whenever  $N$  is a term witnessing  $\perp$ .

Now we have a natural deduction **derivation** with all formulas associated with a term. At each step, the relevant typing context  $\Gamma$  is given by the list of assumptions remaining **undischarged** at that step. Note that  $\Gamma$  is well defined: since we have forbidden assumptions of different **undischarged** assumptions to be assigned the same variable, there won't be any disagreement about the formulas mapped to which a variable is mapped.

We now give some examples of such translations:

Consider the **derivation** of  $\neg\neg(\varphi \vee \neg\varphi)$ , i.e.,  $((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp) \rightarrow \perp$ . Its translation is:

$$\frac{\frac{\frac{[y : (\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp]^2}{\text{in}_1^{\varphi \rightarrow \perp}(x) : \varphi \vee (\varphi \rightarrow \perp)} \quad [x : \varphi]^1}{\frac{1}{\lambda x^\varphi. y(\text{in}_1^{\varphi \rightarrow \perp}(x)) : \varphi \rightarrow \perp}} \quad y(\text{in}_1^{\varphi \rightarrow \perp}(x)) : \perp}{\frac{2}{\lambda y^{(\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp}. y(\text{in}_2^\varphi(\lambda x^\varphi. y(\text{in}_1^{\varphi \rightarrow \perp}(x)))) : ((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp) \rightarrow \perp}} \quad \text{in}_2^\varphi(\lambda x^\varphi. y(\text{in}_1^{\varphi \rightarrow \perp}(x))) : \varphi \vee (\varphi \rightarrow \perp)} \quad y(\text{in}_2^\varphi(\lambda x^\varphi. y(\text{in}_1^{\varphi \rightarrow \perp}(x)))) : \perp$$

The tree has no assumptions, so the context is empty; we get:

$$\vdash \lambda y^{(\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp}. y(\text{in}_2^\varphi(\lambda x^\varphi. y(\text{in}_1^{\varphi \rightarrow \perp}(x)))) : ((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp) \rightarrow \perp$$

If we leave out the last  $\rightarrow$ Intro, the assumptions denoted by  $y$  would be in the context and we would get:

$$y : ((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp) \vdash y(\text{in}_2^\varphi(\lambda x^\varphi. y \text{in}_1^{\varphi \rightarrow \perp}(x))) : \perp$$

Another example:  $\vdash \varphi \rightarrow (\varphi \rightarrow \perp) \rightarrow \perp$

$$\frac{\frac{\frac{[x : \varphi]^2 \quad [y : \varphi \rightarrow \perp]^1}{yx : \perp}}{\lambda y^{\varphi \rightarrow \perp}. yx : (\varphi \rightarrow \perp) \rightarrow \perp}^1}{\lambda x^\varphi. \lambda y^{\varphi \rightarrow \perp}. yx : \varphi \rightarrow (\varphi \rightarrow \perp) \rightarrow \perp}^2$$

Again all assumptions are **discharged** and thus the context is empty, the resulting term is

$$\vdash \lambda x^\varphi. \lambda y^{\varphi \rightarrow \perp}. yx : \varphi \rightarrow (\varphi \rightarrow \perp) \rightarrow \perp$$

If we leave out the last two  $\rightarrow$ Intro inferences, the assumptions denoted by both  $x$  and  $y$  would be in context and we would get

$$x : \varphi, y : \varphi \rightarrow \perp \vdash yx : \perp$$

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## Bibliography