Introduction

Historically the lambda calculus and intuitionistic logic were developed separately. Haskell Curry and William Howard independently discovered a close similarity: types in a typed lambda calculus correspond to formulas in intuitionistic logic in such a way that a derivation of a formula corresponds directly to a typed lambda term with that formula as its type. Moreover, beta reduction in the typed lambda calculus corresponds to certain transformations of derivations.

For instance, a derivation of $\varphi \rightarrow \psi$ corresponds to a term $\lambda x^\varphi . N^\psi$, which has the function type $\varphi \rightarrow \psi$. The inference rules of natural deduction correspond to typing rules in the typed lambda calculus, e.g.,

$\frac{\text{x : } \varphi \Rightarrow N : \psi}{\Rightarrow \lambda x^\varphi . N^\psi : \varphi \rightarrow \psi}$

where the rule on the right means that if $x$ is of type $\varphi$ and $N$ is of type $\psi$, then $\lambda x^\varphi . N$ is of type $\varphi \rightarrow \psi$.

The $\rightarrow$ Elim rule corresponds to the typing rule for composition terms, i.e.,

$\frac{\varphi \rightarrow \psi \quad \varphi \rightarrow \psi}{\Rightarrow P : \varphi \rightarrow \psi \Rightarrow Q : \varphi \Rightarrow P^\varphi \cdot Q^\psi : \psi}$

If a $\rightarrow$ Intro rule is followed immediately by a $\rightarrow$ Elim rule, the derivation can be simplified:

$\frac{\text{x : } \varphi \Rightarrow \psi}{\Rightarrow \lambda x^\varphi . P^\psi : \psi}$

which corresponds to the beta reduction of lambda terms

$(\lambda x^\varphi . P^\psi)Q \Rightarrow P[Q/x]$

Similar correspondences hold between the rules for $\land$ and “product” types, and between the rules for $\lor$ and “sum” types.

This correspondence between terms in the simply typed lambda calculus and natural deduction derivations is called the “Curry-Howard”, or “propositions as types” correspondence. In addition to formulas (propositions) corresponding to types, and proofs to terms, we can summarize the correspondences as follows:
The Curry-Howard correspondence is one of the cornerstones of automated proof assistants and type checkers for programs, since checking a proof witnessing a proposition (as we did above) amounts to checking if a program (term) has the declared type.

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Bibliography