Part I

Intuitionistic Logic

This is a brief introduction to intuitionistic logic produced by Zesen Qian and revised by RZ. It is not yet well integrated with the rest of the text and needs examples and motivations.

Chapter 1

Introduction

1.1 Constructive Reasoning

In constrast to extensions of classical logic by modal operators or second-order quantifiers, intuitionistic logic is "non-classical" in that it restricts classical logic. Classical logic is *non-constructive* in various ways. Intuitionistic logic is intended to capture a more "constructive" kind of reasoning characteristic of a kind of constructive mathematics. The following examples may serve to illustrate some of the underlying motivations.

Suppose someone claimed that they had determined a natural number n with the property that if n is even, the Riemann hypothesis is true, and if n is odd, the Riemann hypothesis is false. Great news! Whether the Riemann hypothesis is true or not is one of the big open questions of mathematics, and they seem to have reduced the problem to one of calculation, that is, to the determination of whether a specific number is prime or not.

What is the magic value of n? They describe it as follows: n is the natural number that is equal to 2 if the Riemann hypothesis is true, and 3 otherwise.

Angrily, you demand your money back. From a classical point of view, the description above does in fact determine a unique value of n; but what you really want is a value of n that is given *explicitly*.

To take another, perhaps less contrived example, consider the following question. We know that it is possible to raise an irrational number to a rational power, and get a rational result. For example, $\sqrt{2}^2 = 2$. What is less clear is whether or not it is possible to raise an irrational number to an *irrational* power, and get a rational result. The following theorem answers this in the affirmative:

Theorem 1.1. *There are irrational numbers a and b such that a^b is rational.*

Proof. Consider $\sqrt{2}^{\sqrt{2}}$. If this is rational, we are done: we can let $a = b = \sqrt{2}$. Otherwise, it is irrational. Then we have

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2,$$

which is rational. So, in this case, let a be $\sqrt{2}^{\sqrt{2}}$, and let b be $\sqrt{2}$. $\square$

Does this constitute a valid proof? Most mathematicians feel that it does. But again, there is something a little bit unsatisfying here: we have proved the existence of a pair of real numbers with a certain property, without being able to say *which* pair of numbers it is. It is possible to prove the same result, but in such a way that the pair a, b *is* given in the proof: take $a = \sqrt{3}$ and $b = \log_3 4$. Then

$$a^b = \sqrt{3}^{\log_3 4} = 3^{1/2 \cdot \log_3 4} = (3^{\log_3 4})^{1/2} = 4^{1/2} = 2,$$

since $3^{\log_3 x} = x$.

Intuitionistic logic is designed to capture a kind of reasoning where moves like the one in the first proof are disallowed. Proving the existence of an x satisfying $\varphi(x)$ means that you have to give a specific x, and a proof that it satisfies φ, like in the second proof. Proving that φ or ψ holds requires that you can prove one or the other.

Formally speaking, intuitionistic logic is what you get if you restrict a proof system for classical logic in a certain way. From the mathematical point of view, these are just formal deductive systems, but, as already noted, they are intended to capture a kind of mathematical reasoning. One can take this to be the kind of reasoning that is justified on a certain philosophical view of mathematics (such as Brouwer's intuitionism); one can take it to be a kind of mathematical reasoning which is more "concrete" and satisfying (along the lines of Bishop's constructivism); and one can argue about whether or not the formal description captures the informal motivation. But whatever philosophical positions we may hold, we can study intuitionistic logic as a formally presented logic; and for whatever reasons, many mathematical logicians find it interesting to do so.

1.2 Syntax of Intuitionistic Logic

The syntax of intuitionistic logic is the same as that for propositional logic. In classical propositional logic it is possible to define connectives by others, e.g., one can define $\varphi \rightarrow \psi$ by $\neg\varphi \lor \psi$, or $\varphi \lor \psi$ by $\neg(\neg\varphi \land \neg\psi)$. Thus, presentations of classical logic often introduce some connectives as abbreviations for these definitions. This is not so in intuitionistic logic, with two exceptions: $\neg\varphi$ can be—and often is—defined as an abbreviation for $\varphi \rightarrow \bot$. Then, of course, $\bot$ must not itself be defined! Also, $\varphi \leftrightarrow \psi$ can be defined, as in classical logic, as $(\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$. int:int:syn:sec

Formulas of propositional intuitionistic logic are built up from *propositional variables* and the propositional constant $\bot$ using *logical connectives*. We have:

1. A denumerable set At_0 of propositional variables p_0, p_1, ...

2. The propositional constant for falsity $\bot$.

3. The logical connectives: $\land$ (conjunction), $\lor$ (disjunction), $\rightarrow$ (conditional)
4. Punctuation marks: (,), and the comma.

int:int:syn: defn:formulas

Definition 1.2 (Formula)**.** The set $\mathrm{Frm}(\mathcal{L}_0)$ of *formulas* of propositional intuitionistic logic is defined inductively as follows:

1. $\bot$ is an atomic formula.
2. Every propositional variable p_i is an atomic formula.
3. If φ and ψ are formulas, then $(\varphi \land \psi)$ is a formula.
4. If φ and ψ are formulas, then $(\varphi \lor \psi)$ is a formula.
5. If φ and ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula.
6. Nothing else is a formula.

In addition to the primitive connectives introduced above, we also use the following *defined* symbols: $\neg$ (negation) and $\leftrightarrow$ (biconditional). Formulas constructed using the defined operators are to be understood as follows:

1. $\neg\varphi$ abbreviates $\varphi \rightarrow \bot$.
2. $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$.

Although $\neg$ is officially treated as an abbreviation, we will sometimes give explicit rules and clauses in definitions for $\neg$ as if it were primitive. This is mostly so we can state practice problems.

1.3 The Brouwer-Heyting-Kolmogorov Interpretation

int:int:bhk: sec

Proofs of validity of intuitionistic propositions using the BHK interpretation are confusing; they have to be explained better.

There is an informal constructive interpretation of the intuitionist connectives, usually known as the Brouwer-Heyting-Kolmogorov interpretation. It uses the notion of a "construction," which you may think of as a constructive proof. (We don't use "proof" in the BHK interpretation so as not to get confused with the notion of a derivation in a formal proof system.) Based on this intuitive notion, the BHK interpretation explains the meanings of the intuitionistic connectives.

1. We assume that we know what constitutes a construction of an atomic statement.

2. A construction of $\varphi_1 \wedge \varphi_2$ is a pair $\langle M_1, M_2 \rangle$ where M_1 is a construction of φ_1 and M_2 is a construction of A_2.

3. A construction of $\varphi_1 \vee \varphi_2$ is a pair $\langle s, M \rangle$ where s is 1 and M is a construction of φ_1, or s is 2 and M is a construction of φ_2.

4. A construction of $\varphi \to \psi$ is a function that converts a construction of φ into a construction of ψ.

5. There is no construction for $\perp$ (absurdity).

6. $\neg\varphi$ is defined as synonym for $\varphi \to \perp$. That is, a construction of $\neg\varphi$ is a function converting a construction of φ into a construction of $\perp$.

Example 1.3. Take $\neg\perp$ for example. A construction of it is a function which, given any construction of $\perp$ as input, provides a construction of $\perp$ as output. Obviously, the identity function Id is such a construction: given a construction M of $\perp$, $\mathrm{Id}(M) = M$ yields a construction of $\perp$.

Generally speaking, $\neg\varphi$ means "A construction of φ is impossible".

Example 1.4. Let us prove $\varphi \to \neg\neg\varphi$ for any proposition φ, which is $\varphi \to ((\varphi \to \perp) \to \perp)$. The construction should be a function f that, given a construction M of φ, returns a construction $f(M)$ of $(\varphi \to \perp) \to \perp$. Here is how f constructs the construction of $(\varphi \to \perp) \to \perp$: We have to define a function g which, when given a construction h of $\varphi \to \perp$ as input, outputs a construction of $\perp$. We can define g as follows: apply the input h to the construction M of φ (that we received earlier). Since the output $h(M)$ of h is a construction of $\perp$, $f(M)(h) = h(M)$ is a construction of $\perp$ if M is a construction of φ.

Example 1.5. Let us give a construction for $\neg(\varphi \wedge \neg\varphi)$, i.e., $(\varphi \wedge (\varphi \to \perp)) \to \perp$. This is a function f which, given as input a construction M of $\varphi \wedge (\varphi \to \perp)$, yields a construction of $\perp$. A construction of a conjunction $\psi_1 \wedge \psi_2$ is a pair $\langle N_1, N_2 \rangle$ where N_1 is a construction of ψ_1 and N_2 is a construction of ψ_2. We can define functions p_1 and p_2 which recover from a construction of $\psi_1 \wedge \psi_2$ the constructions of B_1 and ψ_2, respectively:

$$p_1(\langle N_1, N_2 \rangle) = N_1$$
$$p_2(\langle N_1, N_2 \rangle) = N_2$$

Here is what f does: First it applies p_1 to its input M. That yields a construction of φ. Then it applies p_2 to M, yielding a construction of $\varphi \to \perp$. Such a construction, in turn, is a function $p_2(M)$ which, if given as input a construction of φ, yields a construction of $\perp$. In other words, if we apply $p_2(M)$ to $p_1(M)$, we get a construction of $\perp$. Thus, we can define $f(M) = p_2(p_1(M))$.

Example 1.6. Let us give a construction of $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$, i.e., a function f which turns a construction g of $(\varphi \wedge \psi) \to \chi$ into a construction of $(\varphi \to (\psi \to \chi))$. The construction g is itself a function (from constructions of $\varphi \wedge \psi$ to constructions of C). And the output $f(g)$ is a function h_g from constructions of φ to functions from constructions of ψ to constructions of χ.

Ok, this is confusing. We have to construct a certain function h_g, which will be the output of f for input g. The input of h_g is a construction M of φ. The output of $h_g(M)$ should be a function k_M from constructions N of ψ to constructions of χ. Let $k_{g,M}(N) = g(\langle M, N\rangle)$. Remember that $\langle M, N\rangle$ is a construction of $\varphi \wedge \psi$. So $k_{g,M}$ is a construction of $\psi \to \chi$: it maps constructions N of ψ to constructions of χ. Now let $h_g(M) = k_{g,M}$. That's a function that maps constructions M of φ to constructions $k_{g,M}$ of $\psi \to \chi$. Now let $f(g) = h_g$. That's a function that maps constructions g of $(\varphi \wedge \psi) \to \chi$ to constructions of $\varphi \to (\psi \to \chi)$. Whew!

The statement $\varphi \vee \neg\varphi$ is called the Law of Excluded Middle. We can prove it for some specific φ (e.g., $\bot \vee \neg\bot$), but not in general. This is because the intuitionistic disjunction requires a construction of one of the disjuncts, but there are statements which currently can neither be proved nor refuted (say, Goldbach's conjecture). However, you can't refute the law of excluded middle either: that is, $\neg\neg(\varphi \vee \neg\varphi)$ holds.

Example 1.7. To prove $\neg\neg(\varphi \vee \neg\varphi)$, we need a function f that transforms a construction of $\neg(\varphi \vee \neg\varphi)$, i.e., of $(\varphi \vee (\varphi \to \bot)) \to \bot$, into a construction of $\bot$. In other words, we need a function f such that $f(g)$ is a construction of $\bot$ if g is a construction of $\neg(\varphi \vee \neg\varphi)$.

Suppose g is a construction of $\neg(\varphi \vee \neg\varphi)$, i.e., a function that transforms a construction of $\varphi \vee \neg\varphi$ into a construction of $\bot$. A construction of $\varphi \vee \neg\varphi$ is a pair $\langle s, M\rangle$ where either $s = 1$ and M is a construction of φ, or $s = 2$ and M is a construction of $\neg\varphi$. Let h_1 be the function mapping a construction M_1 of φ to a construction of $\varphi \vee \neg\varphi$: it maps M_1 to $\langle 1, M_2\rangle$. And let h_2 be the function mapping a construction M_2 of $\neg\varphi$ to a construction of $\varphi \vee \neg\varphi$: it maps M_2 to $\langle 2, M_2\rangle$.

Let k be $g \circ h_1$: it is a function which, if given a construction of φ, returns a construction of $\bot$, i.e., it is a construction of $\varphi \to \bot$ or $\neg\varphi$. Now let l be $g \circ h_2$. It is a function which, given a construction of $\neg\varphi$, provides a construction of $\bot$. Since k is a construction of $\neg\varphi$, $l(k)$ is a construction of $\bot$.

Together, what we've done is describe how we can turn a construction g of $\neg(\varphi \vee \neg\varphi)$ into a construction of $\bot$, i.e., the function f mapping a construction g of $\neg(\varphi \vee \neg\varphi)$ to the construction $l(k)$ of $\bot$ is a construction of $\neg\neg(\varphi \vee \neg\varphi)$.

As you can see, using the BHK interpretation to show the intuitionistic validity of formulas quickly becomes cumbersome and confusing. Luckily, there are better derivation systems for intuitionistic logic, and more precise semantic interpretations.

1.4 Natural Deduction

int:int:ntd:sec

Natural deduction without the $\bot_C$ rules is a standard derivation system for intuitionistic logic. We repeat the rules here and indicate the motivation using the BHK interpretation. In each case, we can think of a rule which allows us to conclude that if the premises have constructions, so does the conclusion.

Since natural deduction derivations have undischarged assumptions, we should consider such a derivation, say, of φ from undischarged assumptions Γ, as a function that turns constructions of all $\psi \in \Gamma$ into a construction of φ. If there is a derivation of φ from no undischarged assumptions, then there is a construction of φ in the sense of the BHK interpretation. For the purpose of the discussion, however, we'll suppress the Γ when not needed.

An assumption φ by itself is a derivation of φ from the undischarged assumption φ. This agrees with the BHK-interpretation: the identity function on constructions turns any construction of φ into a construction of φ.

Conjunction

$$\frac{\varphi_1 \qquad \varphi_2}{\varphi_1 \land \varphi_2}\ \land\text{Intro} \qquad\qquad \frac{\varphi_1 \land \varphi_2}{\varphi_i}\ \land\text{Elim}_i \quad i \in \{1, 2\}$$

Suppose we have constructions N_1, N_2 of φ_1 and φ_2, respectively. Then we also have a construction $\varphi_1 \land \varphi_2$, namely the pair $\langle N_1, N_2\rangle$.

A construction of $\varphi_1 \land \varphi_1$ on the BHK interpretation is a pair $\langle N_1, N_2\rangle$. So assume we have such a pair. Then we also have a construction of each conjunct: N_1 is a construction of φ_1 and N_2 is a construction of φ_2.

Conditional

$$\begin{array}{c}[\varphi]^u \\ \vdots \\ \text{u}\ \dfrac{\psi}{\varphi \to \psi}\ \to\text{Intro}\end{array} \qquad\qquad \frac{\varphi \to \psi \qquad \varphi}{\psi}\ \to\text{Elim}$$

If we have a derivation of ψ from undischarged assumption φ, then there is a function f that turns constructions of φ into constructions of ψ. That same function is a construction of $\varphi\to\psi$. So, if the premise of $\to$Intro has a construction conditional on a construction of φ, the conclusion $\varphi\to\psi$ has a construction.

On the other hand, suppose there are constructions N of φ and f of $\varphi \to \psi$. A construction of $\varphi \to \psi$ is a function that turns constructions of φ into constructions of ψ. So, $f(N)$ is a construction of ψ, i.e., the conclusion of $\to$Elim has a construction.

Disjunction

$$\dfrac{\varphi_i}{\varphi_1 \lor \varphi_2}\ \lor\text{Intro}_i \qquad i \in \{1,2\}$$

$$\dfrac{\varphi_1 \lor \varphi_2 \qquad \begin{matrix}[\varphi_1]^u \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\varphi_2]^u \\ \vdots \\ \chi\end{matrix}}{\chi}\ \lor\text{Elim}\ {}_u$$

If we have a construction N_i of φ_i we can turn it into a construction $\langle i, N_i\rangle$ of $\varphi_1 \lor \varphi_2$. On the other hand, suppose we have a construction of $\varphi_1 \lor \varphi_2$, i.e., a pair $\langle i, N_i\rangle$ where N_i is a construction of φ_i, and also functions f_1, f_2, which turn constructions of φ_1, φ_2, respectively, into constructions of χ. Then $f_i(N_i)$ is a construction of χ, the conclusion of $\lor$Elim.

Absurdity

$$\dfrac{\bot}{\varphi}\ \bot_I$$

If we have a derivation of $\bot$ from undischarged assumptions $\psi_1, \ldots, \psi_n$, then there is a function $f(M_1, \ldots, M_n)$ that turns constructions of $\psi_1, \ldots, \psi_n$ into a construction of $\bot$. Since $\bot$ has no construction, there cannot be any constructions of all of $\psi_1, \ldots, \psi_n$ either. Hence, f also has the property that *if* $M_1, \ldots, M_n$ are constructions of $\psi_1, \ldots, \psi_n$, respectively, *then* $f(M_1, \ldots, M_n)$ is a construction of φ.

Rules for $\neg$

Since $\neg\varphi$ is defined as $\varphi \to \bot$, we strictly speaking do not need rules for $\neg$. But if we did, this is what they'd look like:

$${}_n\ \dfrac{\begin{matrix}[\varphi]^n \\ \vdots \\ \bot\end{matrix}}{\neg\varphi}\ \neg\text{Intro} \qquad\qquad \dfrac{\neg\varphi \qquad \varphi}{\bot}\ \neg\text{Elim}$$

Examples of Derivations

1. $\vdash \varphi \to (\neg\varphi \to \bot)$, i.e., $\vdash \varphi \to ((\varphi \to \bot) \to \bot)$

$$
{\scriptstyle 2}\,\dfrac{{\scriptstyle 1}\,\dfrac{\dfrac{[\varphi]^2 \qquad [\varphi \rightarrow \bot]^1}{\bot}\rightarrow\text{Elim}}{(\varphi \rightarrow \bot) \rightarrow \bot}\rightarrow\text{Intro}}{\varphi \rightarrow (\varphi \rightarrow \bot) \rightarrow \bot}\rightarrow\text{Intro}
$$

2. $\vdash ((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$

$$
{\scriptstyle 3}\,\dfrac{{\scriptstyle 2}\,\dfrac{{\scriptstyle 1}\,\dfrac{\dfrac{[(\varphi \wedge \psi) \rightarrow \chi]^3 \qquad \dfrac{[\varphi]^2 \qquad [\psi]^1}{\varphi \wedge \psi}\wedge\text{Intro}}{\chi}\rightarrow\text{Elim}}{\psi \rightarrow \chi}\rightarrow\text{Intro}}{\varphi \rightarrow (\psi \rightarrow \chi)}\rightarrow\text{Intro}}{((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))}\rightarrow\text{Intro}
$$

3. $\vdash \neg(\varphi \wedge \neg\varphi)$, i.e., $\vdash (\varphi \wedge (\varphi \rightarrow \bot)) \rightarrow \bot$

$$
{\scriptstyle 1}\,\dfrac{\dfrac{\dfrac{[\varphi \wedge (\varphi \rightarrow \bot)]^1}{\varphi \rightarrow \bot}\wedge\text{Elim} \qquad \dfrac{[\varphi \wedge (\varphi \rightarrow \bot)]^1}{\varphi}\wedge\text{Elim}}{\bot}\rightarrow\text{Elim}}{(\varphi \wedge (\varphi \rightarrow \bot)) \rightarrow \bot}\rightarrow\text{Intro}
$$

4. $\vdash \neg\neg(\varphi \vee \neg\varphi)$, i.e., $\vdash ((\varphi \vee (\varphi \rightarrow \bot)) \rightarrow \bot) \rightarrow \bot$

$$
{\scriptstyle 2}\,\dfrac{\dfrac{[(\varphi \vee (\varphi \rightarrow \bot)) \rightarrow \bot]^2 \qquad \dfrac{{\scriptstyle 1}\,\dfrac{\dfrac{[(\varphi \vee (\varphi \rightarrow \bot)) \rightarrow \bot]^2 \qquad \dfrac{[\varphi]^1}{\varphi \vee (\varphi \rightarrow \bot)}\vee\text{Intro}}{\bot}\rightarrow\text{Elim}}{\varphi \rightarrow \bot}\rightarrow\text{Intro}}{\varphi \vee (\varphi \rightarrow \bot)}\vee\text{Intro}}{\bot}\rightarrow\text{Elim}}{((\varphi \vee (\varphi \rightarrow \bot)) \rightarrow \bot) \rightarrow \bot}\rightarrow\text{Intro}
$$

Proposition 1.8. *If $\Gamma \vdash \varphi$ in intuitionistic logic, $\Gamma \vdash \varphi$ in classical logic. In particular, if φ is an intuitionistic theorem, it is also a classical theorem.*

Proof. Every natural deduction rule is also a rule in classical natural deduction, so every derivation in intuitionistic logic is also a derivation in classical logic. □

1.5 Axiomatic Derivations

int:int:axd:sec Axiomatic derivations for intuitionistic propositional logic are the conceptually simplest, and historically first, derivation systems. They work just as in classical propositional logic.

Definition 1.9 (Derivability)**.** If Γ is a set of formulas of $\mathcal{L}$ then a *derivation* from Γ is a finite sequence $\varphi_1, \dots, \varphi_n$ of formulas where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. φ_i follows from some φ_j and φ_k with $j < i$ and $k < i$ by modus ponens, i.e., $\varphi_k \equiv \varphi_j \to \varphi_i$.

Definition 1.10 (Axioms)**.** The set of Ax_0 of *axioms* for the intuitionistic propositional logic are all formulas of the following forms:

int:int:axd:ax:land1
$$(\varphi \land \psi) \to \varphi \tag{1.1}$$
int:int:axd:ax:land2
$$(\varphi \land \psi) \to \psi \tag{1.2}$$
int:int:axd:ax:land3
$$\varphi \to (\psi \to (\varphi \land \psi)) \tag{1.3}$$
int:int:axd:ax:lor1
$$\varphi \to (\varphi \lor \psi) \tag{1.4}$$
int:int:axd:ax:lor2
$$\varphi \to (\psi \lor \varphi) \tag{1.5}$$
int:int:axd:ax:lor3
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \lor \psi) \to \chi)) \tag{1.6}$$
int:int:axd:ax:lif1
$$\varphi \to (\psi \to \varphi) \tag{1.7}$$
int:int:axd:ax:lif2
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \tag{1.8}$$
int:int:axd:ax:lfalse1
$$\bot \to \varphi \tag{1.9}$$

Definition 1.11 (Derivability)**.** A formula φ is *derivable* from Γ, written $\Gamma \vdash \varphi$, if there is a derivation from Γ ending in φ.

Definition 1.12 (Theorems)**.** A formula φ is a *theorem* if there is a derivation of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Proposition 1.13. *If $\Gamma \vdash \varphi$ in intuitionistic logic, $\Gamma \vdash \varphi$ in classical logic. In particular, if φ is an intuitionistic theorem, it is also a classical theorem.*

Proof. Every intuitionistic axiom is also a classical axiom, so every derivation in intuitionistic logic is also a derivation in classical logic. □

Chapter 2

Semantics

This chapter collects definitions for semantics for intuitionistic logic. So far only Kripke and topological semantics are covered. There are no examples yet, either of how models make formulas true or of proofs that formulas are valid.

2.1 Introduction

No logic is satisfactorily described without a semantics, and intuitionistic logic is no exception. Whereas for classical logic, the semantics based on valuations is canonical, there are several competing semantics for intuitionistic logic. None of them are completely satisfactory in the sense that they give an intuitionistically acceptable account of the meanings of the connectives. int:sem:int:sec

The semantics based on relational models, similar to the semantics for modal logics, is perhaps the most popular one. In this semantics, propositional variables are assigned to worlds, and these worlds are related by an accessibility relation. That relation is always a partial order, i.e., it is reflexive, antisymmetric, and transitive.

Intuitively, you might think of these worlds as states of knowledge or "evidentiary situations." A state w' is accessible from w iff, for all we know, w' is a possible (future) state of knowledge, i.e., one that is compatible with what's known at w. Once a proposition is known, it can't become un-known, i.e., whenever φ is known at w and Rww', φ is known at w' as well. So "knowledge" is monotonic with respect to the accessibility relation.

If we define "φ is known" as in epistemic logic as "true in all epistemic alternatives," then $\varphi \wedge \psi$ is known at w if in all epistemic alternatives, both φ and ψ are known. But since knowledge is monotonic and R is reflexive, that means that $\varphi \wedge \psi$ is known at w iff φ and ψ are known at w. For the same reason, $\varphi \vee \psi$ is known at w iff at least one of them is known. So for $\wedge$ and $\vee$, the truth conditions of the connectives coincide with those in classical logic.

The truth conditions for the conditional, however, differ from classical logic. $\varphi \to \psi$ is known at w iff at no w' with Rww', φ is known without ψ also being known. This is not the same as the condition that φ is unknown or ψ is known at w. For if we know neither φ nor ψ at w, there might be a future epistemic state w' with Rww' such that at w', φ is known without also coming to know ψ.

We know $\neg\varphi$ only if there is no possible future epistemic state in which we know φ. Here the idea is that if φ were knowable, then in some possible future epistemic state φ becomes known. Since we can't know $\bot$, in that future epistemic state, we would know φ but not know $\bot$.

On this interpretation the principle of excluded middle fails. For there are some φ which we don't yet know, but which we might come to know. For such an φ, both φ and $\neg\varphi$ are unknown, so $\varphi \lor \neg\varphi$ is not known. But we do know, e.g., that $\neg(\varphi \land \neg\varphi)$. For no future state in which we know both φ and $\neg\varphi$ is possible, and we know this independently of whether or not we know φ or $\neg\varphi$.

Relational models are not the only available semantics for intuitionistic logic. The topological semantics is another: here propositions are interpreted as open sets in a topological space, and the connectives are interpreted as operations on these sets (e.g., $\land$ corresponds to intersection).

2.2 Relational models

int:sem:rel: sec In order to give a precise semantics for intuitionistic propositional logic, we have to give a definition of what counts as a model relative to which we can evaluate formulas. On the basis of such a definition it is then also possible to define semantics notions such as validity and entailment. One such semantics is given by relational models.

Definition 2.1. A relational model for intuitionistic propositional logic is a triple $\mathfrak{M} = \langle W, R, V \rangle$, where

1. W is a non-empty set,
2. R is a reflexive and transitive binary relation on W, and
3. V is function assigning to each propositional variable p a subset of W, such that
4. V is monotone with respect to R, i.e., if $w \in V(p)$ and Rww', then $w' \in V(p)$.

int:sem:rel: defn:true-at-w **Definition 2.2.** We define the notion of φ *being true at* w *in* $\mathfrak{M}$, $\mathfrak{M}, w \Vdash \varphi$, inductively as follows:

1. $\varphi \equiv p$: $\mathfrak{M}, w \Vdash \varphi$ iff $w \in V(p)$.
2. $\varphi \equiv \bot$: not $\mathfrak{M}, w \Vdash \varphi$.
3. $\varphi \equiv \neg\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff for no w' such that Rww', $\mathfrak{M}, w' \Vdash \psi$.

4. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$.
5. $\varphi \equiv \psi \vee \chi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$ (or both).
6. $\varphi \equiv \psi \rightarrow \chi$: $\mathfrak{M}, w \Vdash \varphi$ iff for every w' such that Rww', not $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$ (or both).

We write $\mathfrak{M}, w \nVdash \varphi$ if not $\mathfrak{M}, w \Vdash \varphi$. If Γ is a set of formulas, $\mathfrak{M}, w \Vdash \Gamma$ means $\mathfrak{M}, w \Vdash \psi$ for all $\psi \in \Gamma$.

Problem 2.1. Show that according to Definition 2.2, $\mathfrak{M}, w \Vdash \neg\varphi$ iff $\mathfrak{M}, w \Vdash \varphi \rightarrow \bot$.

Proposition 2.3. *Truth at worlds is monotonic with respect to R, i.e., if $\mathfrak{M}, w \Vdash \varphi$ and Rww', then $\mathfrak{M}, w' \Vdash \varphi$.* *int:sem:rel: prop:true-monotonic*

Proof. Exercise. □

Problem 2.2. Prove Proposition 2.3.

2.3 Semantic Notions

int:sem:sem: sec

Definition 2.4. We say φ is *true in the model* $\mathfrak{M} = \langle W, R, V, w_0\rangle$, $\mathfrak{M} \Vdash \varphi$, iff $\mathfrak{M}, w \Vdash \varphi$ for all $w \in W$. φ is *valid*, $\vDash \varphi$, iff it is true in all models. We say a set of formulas Γ *entails* φ, $\Gamma \vDash \varphi$, iff for every model $\mathfrak{M}$ and every w such that $\mathfrak{M}, w \Vdash \Gamma$, $\mathfrak{M}, w \Vdash \varphi$.

Proposition 2.5. *int:sem:sem: prop:sat-entails*

1. *If $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \varphi$, then $\mathfrak{M}, w \Vdash \varphi$.* *int:sem:sem: prop:sat-entails1*
2. *If $\mathfrak{M} \Vdash \Gamma$ and $\Gamma \vDash \varphi$, then $\mathfrak{M} \Vdash \varphi$.* *int:sem:sem: prop:sat-entails2*

Proof.
1. Suppose $\mathfrak{M} \Vdash \Gamma$. Since $\Gamma \vDash \varphi$, we know that if $\mathfrak{M}, w \Vdash \Gamma$, then $\mathfrak{M}, w \Vdash \varphi$. Since $\mathfrak{M}, u \Vdash \Gamma$ for all every $u \in W$, $\mathfrak{M}, w \Vdash \Gamma$. Hence $\mathfrak{M}, w \Vdash \varphi$.
2. Follows immediately from (1).

□

2.4 Topological Semantics

Another way to provide a semantics for intuitionistic logic is using the mathematical concept of a topology. int:sem:top: sec

Definition 2.6. Let X be a set. A *topology on X* is a set $\mathcal{O} \subseteq \wp(X)$ that satisfies the properties below. The elements of $\mathcal{O}$ are called the *open sets* of the topology. The set X together with $\mathcal{O}$ is called a *topological space*.

1. The empty set and the entire space open: $\emptyset$, $X \in \mathcal{O}$.

2. Open sets are closed under finite intersections: if U, $V \in \mathcal{O}$ then $U \cap V \in \mathcal{O}$

3. Open sets are closed under arbitrary unions: if $U_i \in \mathcal{O}$ for all $i \in I$, then $\bigcup\{U_i : i \in I\} \in \mathcal{O}$.

We may write X for a topology if the collection of open sets can be inferred from the context; note that, still, only after X is endowed with open sets can it be called a topology.

Definition 2.7. A *topological model* of intuitionistic propositional logic is a triple $\mathfrak{X} = \langle X, \mathcal{O}, V \rangle$ where $\mathcal{O}$ is a topology on X and V is a function assigning an open set in $\mathcal{O}$ to each propositional variable.

Given a topological model $\mathfrak{X}$, we can define $[\varphi]_{\mathfrak{X}}$ inductively as follows:

1. $V(\bot) = \emptyset$

2. $[p]_{\mathfrak{X}} = V(p)$

3. $[\varphi \wedge \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cap [\psi]_{\mathfrak{X}}$

4. $[\varphi \vee \psi]_{\mathfrak{X}} = [\varphi]_{\mathfrak{X}} \cup [\psi]_{\mathfrak{X}}$

5. $[\varphi \to \psi]_{\mathfrak{X}} = \mathrm{Int}((X \setminus [\varphi]_{\mathfrak{X}}) \cup [\psi]_{\mathfrak{X}})$

Here, $\mathrm{Int}(V)$ is the function that maps a set $V \subseteq X$ to its *interior*, that is, the union of all open sets it contains. In other words,

$$\mathrm{Int}(V) = \bigcup\{U : U \subseteq V \text{ and } U \in \mathcal{O}\}.$$

Note that the interior of any set is always open, since it is a union of open sets. Thus, $[\varphi]_{\mathfrak{X}}$ is always an open set.

Although topological semantics is highly abstract, there are ways to think about it that might motivate it. Suppose that the elements, or "points," of X are points at which statements can be evaluated. The set of all points where φ is true is the proposition expressed by φ. Not every set of points is a potential proposition; only the elements of $\mathcal{O}$ are. $\varphi \vDash \psi$ iff ψ is true at every point at which φ is true, i.e., $[\varphi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$, for all X. The absurd statement $\bot$ is never true, so $[\bot]_{\mathfrak{X}} = \emptyset$. How must the propositions expressed by $\psi \wedge \chi$, $\psi \vee \chi$, and $\psi \to \chi$ be related to those expressed by ψ and χ for the intuitionistically valid laws to hold, i.e., so that $\varphi \vdash \psi$ iff $[\varphi]_{\mathfrak{X}} \subset [\psi]_{\mathfrak{X}}$. $\bot \vdash \varphi$ for any φ, and only $\emptyset \subseteq U$ for all U. Since $\psi \wedge \chi \vdash \psi$, $[\psi \wedge \chi]_{\mathfrak{X}} \subseteq [\psi]_{\mathfrak{X}}$, and similarly $[\psi \wedge \chi]_{\mathfrak{X}} \subseteq [\chi]_{\mathfrak{X}}$. The largest set satisfying $W \subseteq U$ and $W \subseteq V$ is $U \cap V$. Conversely, $\psi \vdash \psi \vee \chi$ and $\chi \vdash \psi \vee \chi$, and so $[\psi]_{\mathfrak{X}} \subseteq [\psi \vee \chi]_{\mathfrak{X}}$ and $[\chi]_{\mathfrak{X}} \subseteq [\psi \vee \chi]_{\mathfrak{X}}$. The smallest set W such that $U \subseteq W$ and $V \subseteq W$ is $U \cup V$. The definition for $\to$ is tricky: $\varphi \to \psi$ expresses the weakest proposition that, combined with φ, entails ψ. That $\varphi \to \psi$ combined with φ entails ψ is clear from $(\varphi \to \psi) \wedge \varphi \vdash \psi$. So $[\varphi \to \psi]_{\mathfrak{X}}$ should be the greatest open set such that $[\varphi \to \psi]_{\mathfrak{X}} \cap [\varphi]_{\mathfrak{X}} \subset [\psi]_{\mathfrak{X}}$, leading to our definition.

Chapter 3

Soundness and Completeness

This chapter collects soundness and completeness results for propositional intuitionistic logic. It needs an introduction. The completeness proof makes use of facts about provability that should be stated and proved explicitly somehwere.

3.1 Soundness of Axiomatic Derivations

int:sc:sax:sec

The soundness proof relies on the fact that all axioms are intuitionistically valid; this still needs to be proved, e.g., in the Semantics chapter.

Theorem 3.1 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.* *int:sc:sax:thm:soundness*

Proof. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the number n of formulas in the derivation of φ from Γ. We show that if φ_1, …, $\varphi_n = \varphi$ is a derivation from Γ, then $\Gamma \vDash \varphi_n$. Note that if φ_1, …, φ_n is a derivation, so is φ_1, …, φ_k for any $k < n$.

There are no derivations of length 0, so for $n = 0$ the claim holds vacuously. So the claim holds for all derivations of length $< n$. We distinguish cases according to the justification of φ_n.

1. φ_n is an axiom. All axioms are valid, so $\Gamma \vDash \varphi_n$ for any Γ.

2. $\varphi_n \in \Gamma$. Then for any $\mathfrak{M}$ and w, if $\mathfrak{M}, w \Vdash \Gamma$, obviously $\mathfrak{M} \Vdash \Gamma \varphi_n[w]$, i.e., $\Gamma \vDash \varphi$.

3. φ_n follows by MP from φ_i and $\varphi_j \equiv \varphi_i \to \varphi_n$. φ_1, …, φ_i and φ_1, …, φ_j are derivations from Γ, so by inductive hypothesis, $\Gamma \vDash \varphi_i$ and $\Gamma \vDash \varphi_i \to \varphi_n$.

Suppose $\mathfrak{M}, w \Vdash \Gamma$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \varphi_i \to \varphi_n$, $\mathfrak{M}, w \Vdash \varphi_i \to \varphi_n$. By definition, this means that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \varphi_i$ then $\mathfrak{M}, w' \Vdash \varphi_n$. Since R is reflexive, w is among the w' such that Rww', i.e., we have that if $\mathfrak{M}, w \Vdash \varphi_i$ then $\mathfrak{M}, w \Vdash \varphi_n$. Since $\Gamma \vDash \varphi_i$, $\mathfrak{M}, w \Vdash \varphi_i$. So, $\mathfrak{M}, w \Vdash \varphi_n$, as we wanted to show.

□

3.2 Soundness of Natural Deduction

int:sc:snd:sec

int:sc:snd:thm:soundness

Theorem 3.2 (Soundness). *If $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.*

Proof. We prove that if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$. The proof is by induction on the derivation of φ from Γ.

1. If the derivation consists of just the assumption φ, we have $\varphi \vdash \varphi$, and want to show that $\varphi \vDash \varphi$. Consider any model $\mathfrak{M}$ such that $\mathfrak{M} \Vdash \varphi$. Then trivially $\mathfrak{M} \Vdash \varphi$.

2. The derivation ends in $\wedge$Intro: The derivations of the premises ψ from undischarged assumptions Γ and of χ from undischarged assumptions Δ show that $\Gamma \vdash \psi$ and $\Delta \vdash \chi$. By induction hypothesis we have that $\Gamma \vDash \psi$ and $\Gamma \vDash \chi$. We have to show that $\Gamma \cup \Delta \vDash \varphi \wedge \psi$, since the undischarged assumptions of the entire derivation are Γ together with Δ. So suppose $\mathfrak{M} \Vdash \Gamma \cup \Delta$. Then also $\mathfrak{M} \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M} \Vdash \psi$. Similarly, $\mathfrak{M} \Vdash \chi$. So $\mathfrak{M} \Vdash \psi \wedge \chi$.

3. The derivation ends in $\wedge$Elim: The derivation of the premise $\psi \wedge \chi$ from undischarged assumptions Γ shows that $\Gamma \vdash \psi \wedge \chi$. By induction hypothesis, $\Gamma \vDash \psi \wedge \chi$. We have to show that $\Gamma \vDash \psi$. So suppose $\mathfrak{d} \Vdash$. Since $\Gamma \vDash \psi \wedge \chi$, $\mathfrak{M} \Vdash \psi \wedge \chi$. Then also $\mathfrak{M} \Vdash \psi$. Similarly if $\wedge$Elim ends in χ, then $\Gamma \vDash \chi$.

4. The derivation ends in $\vee$Intro: Suppose the premise is ψ, and the undischarged assumptions of the derivation ending in ψ are Γ. Then we have $\Gamma \vdash \psi$ and by inductive hypothesis, $\Gamma \vDash B$. We have to show that $\Gamma \vDash \psi \vee \chi$. Suppose $\mathfrak{M} \Vdash \Gamma$. Since $\Gamma \vDash \psi$, $\mathfrak{M} \Vdash \psi$. But then also $\mathfrak{M} \Vdash \psi \vee \chi$. Similarly, if the premise is χ, we have that $\Gamma \vDash \chi$.

5. The derivation ends in $\vee$Elim: The derivations ending in the premises are of $\psi \vee \chi$ from undischarged assumptions Γ, of θ from undischarged assumptions $\Delta_1 \cup \{\psi\}$, and of θ from undischarged assumptions $\Delta_2 \cup \{\chi\}$. So we have $\Gamma \vdash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vdash \theta$, and $\Delta_2 \cup \{\chi\} \vdash \theta$. By induction hypothesis, $\Gamma \vDash \psi \vee \chi$, $\Delta_1 \cup \{\psi\} \vDash \theta$, and $\Delta_2 \cup \{\chi\} \vDash \theta$. We have to prove that $\Gamma \cup \Delta_1 \cup \Delta_2 \vDash \theta$.

Suppose $\mathfrak{M} \Vdash \Gamma \cup \Delta_1 \cup \Delta_2$. Then $\mathfrak{M} \Vdash \Gamma$ and since $\Gamma \vDash \psi \lor \chi$, $\mathfrak{M} \Vdash \psi \lor \chi$. By definition of $\mathfrak{M} \Vdash$, either $\mathfrak{M} \Vdash \psi$ or $\mathfrak{M} \Vdash \chi$. So we distinguish cases: (a) $\mathfrak{M} \Vdash \psi$. Then $\mathfrak{M} \Vdash \Delta_1 \cup \{\psi\}$. Since $\Delta_1 \cup \psi \vDash \theta$, we have $\mathfrak{M} \Vdash \theta$. (b) $\mathfrak{M} \Vdash \chi$. Then $\mathfrak{M} \Vdash \Delta_2 \cup \{\chi\}$. Since $\Delta_2 \cup \chi \vDash \theta$, we have $\mathfrak{M} \Vdash \theta$. So in either case, $\mathfrak{M} \Vdash \theta$, as we wanted to show.

6. The derivation ends with $\rightarrow$Intro concluding $\psi \rightarrow \chi$. Then the premise is χ, and the derivation ending in the premise has undischarged assumptions $\Gamma \cup \{\psi\}$. So we have that $\Gamma \cup \{\psi\} \vdash \chi$, and by induction hypothesis that $\Gamma \cup \{\psi\} \vDash \chi$. We have to show that $\Gamma \vDash \psi \rightarrow \chi$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma$. We want to show that that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \psi$, then $\mathfrak{M}, w' \Vdash \chi$. So assume that Rww' and $\mathfrak{M}, w' \Vdash \psi$. By Proposition 2.3, $\mathfrak{M}, w' \Vdash \Gamma$. Since $\Gamma \cup \{\psi\} \vDash \chi$, $\mathfrak{M}, w' \Vdash \chi$, which is what we wanted to show.

7. The derivation ends in $\rightarrow$Elim and conclusion χ. The premises are $\psi \rightarrow \chi$ amd ψ, with derivations from undischarged assumptions Γ, Δ. So we have $\Gamma \vdash \psi \rightarrow \chi$ and $\Delta \vdash \psi$. By inductive hypothesis, $\Gamma \vDash \psi \rightarrow \chi$ and $\Delta \vDash \psi$. We have to show that $\Gamma \cup \Delta \vDash \chi$.

   Suppose $\mathfrak{M}, w \Vdash \Gamma \cup \Delta$. Since $\mathfrak{M}, w \Vdash \Gamma$ and $\Gamma \vDash \psi \rightarrow \chi$, $\mathfrak{M}, w \Vdash \psi \rightarrow \chi$. By definition, this means that for all w' such that Rww', if $\mathfrak{M}, w' \Vdash \psi$ then $\mathfrak{M}, w' \Vdash \chi$. Since R is reflexive, w is among the w' such that Rww', i.e., we have that if $\mathfrak{M}, w \Vdash \psi$ then $\mathfrak{M}, w \Vdash \chi$. Since $\mathfrak{M}, w \Vdash \Delta$ and $\Delta \vDash \psi$, $\mathfrak{M}, w \Vdash \psi$. So, $\mathfrak{M}, w \Vdash \chi$, as we wanted to show.

8. The derivation ends in $\bot_I$, concluding φ. The premise is $\bot$ and the undischarged assumptions of the derivation of the premise are Γ. Then $\Gamma \vdash \bot$. By inductive hypothesis, $\Gamma \vDash \bot$. We have to show $\Gamma \vDash \varphi$.

   We proceed indirectly. If $\Gamma \nvDash \varphi$ there is a model $\mathfrak{M}$ and world w such that $\mathfrak{M}, w \Vdash \Gamma$ and $\mathfrak{M}, w \nVdash \varphi$. Since $\Gamma \vDash \bot$, $\mathfrak{M}, w \Vdash \bot$. But that's impossible, since by definition, $\mathfrak{M}, w \nVdash \bot$. So $\Gamma \vDash \varphi$.

9. The derivation endss in $\neg$Intro: Exercise.

10. The derivation ends in $\neg$Elim: Exercise.

□

Problem 3.1. Complete the proof of Theorem 3.2. For the cases for $\neg$Intro and $\neg$Elim, use the definition of $\mathfrak{M}, w \Vdash \neg\varphi$ in Definition 2.2, i.e., don't treat $\neg\varphi$ as defined by $\varphi \rightarrow \bot$.

3.3 Lindenbaum's Lemma

int:sc:lin:sec

Definition 3.3. A set of formulas Γ is *prime* iff int:sc:lin:defn:prime

int:sc:lin: defn:prime1
1. Γ is consistent.

int:sc:lin: defn:prime2
2. If $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$, and

int:sc:lin: defn:prime3
3. If $\varphi \lor \psi \in \Gamma$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$.

int:sc:lin: lem:lindenbaum
Lemma 3.4 (Lindenbaum's Lemma). *If $\Gamma \nvdash \varphi$, there is a $\Gamma^* \supseteq \Gamma$ such that Γ^* is prime and $\Gamma^* \nvdash \varphi$.*

Proof. Let $\psi_1 \lor \chi_1$, $\psi_2 \lor \chi_2$, ..., be an enumeration of all formulas of the form $\psi \lor \chi$. We'll define an increasing sequence of sets of formulas Γ_n, where each Γ_{n+1} is defined as Γ_n together with one new formula. Γ^* will be the union of all Γ_n. The new formulas are selected so as to ensure that Γ^* is prime and still $\Gamma^* \nvdash \varphi$. This means that at each step we should find the first disjunction $\psi_i \lor \chi_i$ such that:

1. $\Gamma_n \vdash \psi_i \lor \chi_i$
2. $\psi_i \notin \Gamma_n$ and $\chi_i \notin \Gamma_n$

We add to Γ_n either ψ_i if $\Gamma_n \cup \{\psi_i\} \nvdash \varphi$, or χ_i otherwise. We'll have to show that this works. For now, let's define $i(n)$ as the least i such that (1) and (2) hold.

Define $\Gamma_0 = \Gamma$ and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_{i(n)}\} & \text{if } \Gamma_n \cup \{\psi_{i(n)}\} \nvdash \varphi \\ \Gamma_n \cup \{\chi_{i(n)}\} & \text{otherwise} \end{cases}$$

If $i(n)$ is undefined, i.e., whenever $\Gamma \vdash \psi \lor \chi$, either $\psi \in \gamma_n$ or $\chi \in \Gamma_n$, we let $\Gamma_{n+1} = \Gamma_n$. Now let $\Gamma^* = \bigcup_{n=0}^{\infty} \Gamma_n$

First we show that for all n, $\Gamma_n \nvdash \varphi$. We proceed by induction on n. For $n = 0$ the claim holds by the hypothesis of the theorem, i.e., $\Gamma \nvdash \varphi$. If $n > 0$, we have to show that if $\Gamma_n \nvdash \varphi$ then $\Gamma_{n+1} \nvdash \varphi$. If $i(n)$ is undefined, $\Gamma_{n+1} = \Gamma_n$ and there is nothing to prove. So suppose $i(n)$ is defined. For simplicity, let $i = i(n)$.

We'll prove the contrapositive of the claim. Suppose $\Gamma_{n+1} \vdash \varphi$. By construction, $\Gamma_{n+1} = \Gamma_n \cup \{\psi_i\}$ if $\Gamma_n \cup \{\psi_i\} \nvdash \varphi$, or else $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. It clearly can't be the first, since then $\Gamma_{n+1} \nvdash \varphi$. Hence, $\Gamma_n \cup \{\psi_i\} \vdash \varphi$ and $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\}$. By definition of $i(n)$, we have that $\Gamma_n \vdash \psi_i \lor \chi_i$. We have $\Gamma_n \cup \{\psi_i\} \vdash \varphi$. We also have $\Gamma_{n+1} = \Gamma_n \cup \{\chi_i\} \vdash \varphi$. Hence, $\Gamma_n \vdash \varphi$, which is what we wanted to show.

If $\Gamma^* \vdash \varphi$, there would be some finite subset $\Gamma' \subseteq \Gamma^*$ such that $\Gamma' \vdash \varphi$. Each $\theta \in \Gamma'$ must be in Γ_i for some i. Let n be the largest of these. Since $\Gamma_i \subseteq \Gamma_n$ if $i \leq n$, $\Gamma' \subseteq \Gamma_n$. But then $\Gamma_n \vdash \varphi$, contrary to our proof above that $\Gamma_n \nvdash \varphi$.

Lastly, we show that Γ^* is prime, i.e., satisfies conditions (1), (2), and (3) of Definition 3.3.

First, $\Gamma^* \nvdash \varphi$, so Γ^* is consistent, so (1) holds.

We now show that if $\Gamma^* \vdash \psi \lor \chi$, then either $\psi \in \Gamma^*$ or $\chi \in \Gamma^*$. This proves (3), since if $\psi \in \Gamma^*$ then also $\Gamma^* \vdash \psi$, and similarly for χ. So assume $\Gamma^* \vdash \psi \lor \chi$ but $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$. Since $\Gamma^* \vdash \psi \lor \chi$, $\Gamma_n \vdash \psi \lor \chi$ for some n. $\psi \lor \chi$ appears on the enumeration of all disjunctions, say as $\psi_j \lor \chi_j$. $\psi_j \lor \chi_j$ satisfies the properties in the definition of $i(n)$, namely we have $\Gamma_n \vdash \psi_j \lor \chi_j$, while $\psi_j \notin \Gamma_n$ and $\chi_j \notin \Gamma_n$. At each stage, at least one fewer disjunction $\psi_i \lor \chi_i$ satisfies the conditions (since at each stage we add either ψ_i or χ_i), so at some stage m we will have $j = i(\Gamma_m)$. But then either $\psi \in \Gamma_{m+1}$ or $\chi \in \Gamma_{m+1}$, contrary to the assumption that $\psi \notin \Gamma^*$ and $\chi \notin \Gamma^*$.

Now suppose $\Gamma^* \vdash \varphi$. Then $\Gamma^* \vdash \varphi \lor \varphi$. But we've just proved that if $\Gamma^* \vdash \varphi \lor \varphi$ then $\varphi \in \Gamma^*$. Hence, Γ^* satisfies (1) of Definition 3.3. □

3.4 The Canonical Model

The worls in our model will be finite sequences σ of natural numbers, i.e., $\sigma \in \mathbb{N}^*$. Note that $\mathbb{N}^*$ is inductively defined by: int:sc:mod:sec

1. $\Lambda \in \mathbb{N}^*$.
2. If $\sigma \in \mathbb{N}^*$ and $n \in \Sigma$, then $\sigma.n \in \mathbb{N}^*$ (where $\sigma.n$ is $\sigma \frown \langle n \rangle$).
3. Nothing else is in $\mathbb{N}^*$.

So we can use $\mathbb{N}^*$ to give inductive definitions.

Let $\langle \psi_1, \chi_1 \rangle$, $\langle \psi_2, \chi_s \rangle$, ..., be an enumeration of all pairs of formulas. Given a set of formulas Δ, define $\Delta(\sigma)$ by induction as follows:

1. $\Delta(\Lambda) = \Delta$
2. $\Delta(\sigma.n) =$

$$\begin{cases} (\Delta(\sigma) \cup \{\psi_n\})^* & \text{if } \Delta(\sigma) \cup \{\psi_n\} \nvdash \chi_n \\ \Delta(\sigma) & \text{otherwise} \end{cases}$$

Here by $(\Delta(\sigma) \cup \{\psi_n\})^*$ we mean the prime set of formulas which exists by Lemma 3.4 applied to the set $\Delta(\sigma) \cup \{\psi_n\}$. Note that by this definition, if $\Delta(\sigma) \cup \{\psi_n\} \nvdash \chi_n$, then $\Delta(\sigma.n) \vdash \psi_n$ and $\Delta(\sigma.n) \nvdash \chi_n$. Note also that $\Delta(\sigma) \subseteq \Delta(\sigma.n)$ for any n. If Δ is prime, then $\Delta(\sigma)$ is prime for all σ.

Definition 3.5. Suppose Δ is prime. Then the *canonical model* for Δ is defined by: int:sc:mod:defn:canonical-model

1. $W = \mathbb{N}^*$, the set of finite sequences of natural numbers.
2. R is the partial order according to which $R\sigma\sigma'$ iff σ is an initial segment of σ' (i.e., $\sigma' = \sigma \frown \sigma''$ for some sequence σ'').
3. $V(p) = \{\sigma : p \in \Delta(\sigma)\}$.

It is easy to verify that R is indeed a partial order. Also, the monotonicity condition on V is satisfied. Since $\Delta(\sigma) \subseteq \Delta(\sigma.n)$ we get $\Delta(\sigma) \subseteq \Delta(\sigma')$ whenever $R\sigma\sigma'$ by induction on σ.

3.5 The Truth Lemma

int:sc:tru:sec

int:sc:tru:lem:truth **Lemma 3.6.** *If Δ is prime, then $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\Delta(\sigma) \vdash \varphi$.*

Proof. By induction on φ.

1. $\varphi \equiv \bot$: Since $\Delta(\sigma)$ is prime, it is consistent, so $\Delta(\sigma) \nvdash \varphi$. By definition, $\mathfrak{M}(\Delta), \sigma \nVdash \varphi$.

2. $\varphi \equiv p$: By definition of $\Vdash$, $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\sigma \in V(p)$, i.e., $\Delta(\sigma) \vdash \varphi$.

3. $\varphi \equiv \neg\psi$: exercise.

4. $\varphi \equiv \psi \wedge \chi$: $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. By induction hypothesis, $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ iff $\Delta(\sigma) \vdash \psi$, and similarly for χ. But $\Delta(\sigma) \vdash \psi$ and $\Delta(\sigma) \vdash \chi$ iff $\Delta(\sigma) \vdash \varphi$.

5. $\varphi \equiv \psi \vee \chi$: $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$ iff $\mathfrak{M}(\Delta), \sigma \Vdash \psi$ or $\mathfrak{M}(\Delta), \sigma \Vdash \chi$. By induction hypothesis, this holds iff $\Delta(\sigma) \vdash \psi$ of $\Delta(\sigma) \vdash \chi$. We have to show that this in turn holds iff $\Delta(\sigma) \vdash \varphi$. The left-to-right direction is clear. The right-to-left direction follows since $\Delta(\sigma)$ is prime.

6. $\varphi \equiv \psi \rightarrow \chi$: First the contrapositive of the left-to-right direction: Assume $\Delta(\sigma) \nvdash \psi \rightarrow \chi$. Then also $\Gamma * (\sigma) \cup \{\psi\} \nvdash \chi$. Since $\langle \psi, \chi \rangle$ is $\langle \psi_n, \chi_n \rangle$ for some n, we have $\Delta(\sigma.n) = (\Delta(\sigma) \cup \{\psi\})^*$, and $\Delta(\sigma.n) \vdash \psi$ but $\nvdash \chi$. By inductive hypothesis, $\mathfrak{M}(\Delta), \sigma.n \Vdash \psi$ and $\mathfrak{M}(\Delta), \sigma.n \nVdash \chi$. Since $R\sigma(\sigma.n)$, this means that $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

   Now assume $\Delta(\sigma) \vdash \psi \rightarrow \chi$, and let $R\sigma\sigma'$. Since $\Delta(\sigma) \subseteq \Delta(\sigma')$, we have: if $\Delta(\sigma') \vdash \psi$, then $\Delta(\sigma') \vdash \chi$. In other words, for every σ' such that $R\sigma\sigma'$, either $\Delta(\sigma') \nvdash \psi$ or $\Delta(\sigma') \vdash \chi$. By induction hypothesis, this means that whenever $R\sigma\sigma'$, either $\mathfrak{M}(\Delta), \sigma' \nVdash \psi$ or $\mathfrak{M}(\Delta), \sigma' \Vdash \chi$, i.e., $\mathfrak{M}(\Delta), \sigma \Vdash \varphi$.

□

3.6 The Completeness Theorem

int:sc:cpl:sec

int:sc:cpl:thm:completeness **Theorem 3.7.** *If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.*

Proof. We prove the contrapositive: Suppose $\Gamma \nvdash \varphi$. Then by Lemma 3.4, there is a prime set $\Gamma^* \supseteq \Gamma$ such that $\Gamma^* \nvdash \varphi$. Consider the canonical model $\mathfrak{M}(\Gamma^*)$ for Γ^* as defined in Definition 3.5. For any $\psi \in \Gamma$, $\Gamma^* \vdash \psi$. Note that $\Gamma * (\Lambda) = \Gamma *$. By the Truth Lemma (Lemma 3.6), we have $\mathfrak{M}(\Gamma^*), \Lambda \Vdash \psi$ for all $\psi \in \Gamma$ and $\mathfrak{M}(\Gamma^*), \Lambda \nVdash \varphi$. This shows that $\Gamma \nvDash \varphi$. □

Chapter 4

Propositions as Types

This is a *very experimental* draft of a chapter on the Curry-Howard correspondence. It needs more explanation and motivation, and there are probably errors and omissions. The proof of normalization should be reviewed and expanded. There are no examples for the product type. Permuation and simplification conversions are not covered. It will make a lot more sense once there is also material on the (typed) lambda calculus which is basically presupposed here. Use with extreme caution.

4.1 Introduction

Historically the lambda calculus and intuitionistic logic were developed separately. Haskell Curry and William Howard independently discovered a close similarity: types in a typed lambda calculus correspond to formulas in intuitionistic logic in such a way that a derivation of a formula corresponds directly to a typed lambda term with that formula as its type. Moreover, beta reduction in the typed lambda calculus corresponds to certain transformations of derivations. int:pty:int:sec

For instance, a derivation of $\varphi \to \psi$ corresponds to a term $\lambda x^{\varphi}.\, N^{\psi}$, which has the function type $\varphi \to \psi$. The inference rules of natural deduction correspond to typing rules in the typed lambda calculus, e.g.,

$$\dfrac{\begin{matrix}[\varphi]^x \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}\; x\ {\to}\mathrm{Intro} \qquad \text{corresponds to} \qquad \dfrac{x : \varphi \;\Rightarrow\; N : \psi}{\Rightarrow\; \lambda x^{\varphi}.\, N^{\psi} : \varphi \to \psi}\;\lambda$$

where the rule on the right means that if x is of type φ and N is of type ψ, then $\lambda x^{\varphi}.\, N$ is of type $\varphi \to \psi$.

The $\to$Elim rule corresponds to the typing rule for composition terms, i.e.,

$$\dfrac{\varphi \to \psi \qquad \varphi}{\psi}\to\text{Elim} \qquad \text{corresponds to}$$

$$\dfrac{\Rightarrow\ P : \varphi \to \psi \qquad \Rightarrow\ Q : \varphi}{\Rightarrow\ P^{\varphi\to\psi}Q^{\varphi} : \psi}\text{app}$$

If a $\to$Intro rule is followed immediately by a $\to$Elim rule, the derivation can be simplified:

$$\dfrac{{}_x\dfrac{\begin{matrix}[\varphi]^x \\ \vdots \\ \psi\end{matrix}}{\varphi \to \psi}\to\text{Intro} \qquad \begin{matrix}\vdots \\ \varphi\end{matrix}}{\psi}\to\text{Elim} \qquad \rhd_1 \qquad \begin{matrix}\vdots \\ \varphi \\ \vdots \\ \psi\end{matrix}$$

which corresponds to the beta reduction of lambda terms

$$(\lambda x^{\varphi}.\, P^{\psi})Q \quad \rhd_1 \quad P[Q/x].$$

Similar correspondences hold between the rules for $\land$ and "product" types, and between the rules for $\lor$ and "sum" types.

This correspondence between terms in the simply typed lambda calculus and natural deduction derivations is called the "Curry-Howard", or "propositions as types" correspondence. In addition to formulas (propositions) corresponding to types, and proofs to terms, we can summarize the correspondences as follows:

logic	program
proposition	type
proof	term
assumption	variable
discharged assumption	bind variable
not discharged assumption	free variable
implication	function type
conjunction	product type
disjunction	sum type
absurdity	bottom type

The Curry-Howard correspondence is one of the cornerstones of automated proof assistants and type checkers for programs, since checking a proof witnessing a proposition (as we did above) amounts to checking if a program (term) has the declared type.

4.2 Sequent Natural Deduction

int:pty:snd:sec Let us write $\Gamma \Rightarrow \varphi$ if there is a natural deduction derivation with Γ as undischarged assumptions and φ as conclusion; or $\Rightarrow \varphi$ if Γ is empty.

We write $\Gamma, \varphi_1, \dots, \varphi_n$ for $\Gamma \cup \{\varphi_1, \dots, \varphi_n\}$, and Γ, Δ for $\Gamma \cup \Delta$.

Observe that when we have $\Gamma \Rightarrow \varphi \wedge \varphi$, meaning we have a derivation with Γ as undischarged assumptions and $\varphi \wedge \varphi$ as end-formula, then by applying $\wedge$Elim at the bottom, we can get a derivation with the same undischarged assumptions and φ as conclusion. In other words, if $\Gamma \Rightarrow \varphi \wedge \psi$, then $\Gamma \Rightarrow \varphi$.

$$\frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi}\ \wedge\mathrm{Elim} \qquad\qquad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi}\ \wedge\mathrm{Elim}$$

The label $\wedge$Elim hints at the relation with the rule of the same name in natural deduction.

Likewise, suppose we have $\Gamma, \varphi \Rightarrow \psi$, meaning we have a derivation with undischarged assumptions Γ, φ and end-formula ψ. If we apply the $\to$Intro rule, we have a derivation with Γ as undischarged assumptions and $\varphi \to \psi$ as the end-formula, i.e., $\Gamma \Rightarrow \varphi \to \psi$. Note how this has made the discharge of assumptions more explicit.

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ \to\mathrm{Intro}$$

We can draw conclusions from other rules in the same fashion, which is spelled out as follows:

$$\frac{\Gamma \Rightarrow \varphi \qquad \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \varphi \wedge \psi}\ \wedge\mathrm{Intro}$$

$$\frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi}\ \wedge\mathrm{Elim}_1 \qquad \frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi}\ \wedge\mathrm{Elim}_2$$

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi}\ \vee\mathrm{Intro}_1 \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}\ \vee\mathrm{Intro}_2$$

$$\frac{\Gamma \Rightarrow \varphi \vee \psi \qquad \Delta, \varphi \Rightarrow \chi \qquad \Delta', \psi \Rightarrow \chi}{\Gamma, \Delta, \Delta' \Rightarrow \chi}\ \vee\mathrm{Elim}$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ \to\mathrm{Intro} \qquad \frac{\Delta \Rightarrow \varphi \to \psi \qquad \Gamma \Rightarrow \varphi}{\Gamma, \Delta \Rightarrow \psi}\ \to\mathrm{Elim}$$

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}\ \bot_I$$

Any assumption by itself is a derivation of φ from φ, i.e., we always have $\varphi \Rightarrow \varphi$.

$$\frac{}{\varphi \Rightarrow \varphi}$$

Together, these rules can be taken as a calculus about what natural deduction derivations exist. They can also be taken as a notational variant of natural deduction, in which each step records not only the formula derived but also the undischarged assumptions from which it was derived.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi}{\varphi \Rightarrow \varphi \lor (\varphi \to \bot)} \qquad \psi \Rightarrow \psi}{\varphi, \psi \to\ \Rightarrow \bot}}{(\psi \Rightarrow \varphi \to \bot}}{(\psi \Rightarrow \varphi \lor (\varphi \to \bot)} \qquad (\psi \Rightarrow \psi}{(\psi \Rightarrow \bot}}{\Rightarrow \psi \to \bot}
$$

where ψ is short for $(\varphi \lor (\varphi \to \bot)) \to \bot$.

4.3 Proof Terms

int:pty:ter:sec

We give the definition of proof terms, and then establish its relation with natural deduction derivations.

Definition 4.1 (Proof terms)**.** Proof terms are inductively generated by the following rules:

1. A single variable x is a proof term.
2. If P and Q are proof terms, then PQ is also a proof term.
3. If x is a variable, φ is a formula, and N is a proof term, then $\lambda x^{\varphi}. N$ is also a proof term.
4. If P and Q are proof terms, then $\langle P, Q \rangle$ is a proof term.
5. If M is a proof term, then $\mathrm{p}_i(M)$ is also a proof term, where i is 1 or 2.
6. If M is a proof term, and φ is a formula, then $\mathrm{in}_i^{\varphi}(M)$ is a proof term, where i is 1 or 2.
7. If M, N_1, N_2 is proof terms, and x_1, x_2 are variables, then $\mathrm{case}(M, x_1.N_1, x_2.N_2)$ is a proof term.
8. If M is a proof term and φ is a formula, then $\mathrm{contr}_{\varphi}(M)$ is proof term.

Each of the above rules corresponds to an inference rule in natural deduction. Thus we can inductively assign proof terms to the formulas in a derivation. To make this assignment unique, we must distinguish between the two versions of $\land$Elim and of $\lor$Intro. For instance, the proof terms assigned to the conclusion of $\lor$Intro must carry the information whether $\varphi \lor \psi$ is inferred from φ or from ψ. Suppose M is the term assigned to $\varphi$from which $\varphi \lor \psi$ is inferred. Then the proof term assigned to $\varphi \lor \psi$ is $\mathrm{in}_1^{\varphi}(M)$. If we instead infer $\psi \lor \varphi$ then the proof term assigned is $\mathrm{in}_2^{\varphi}(M)$.

The term $\lambda x^{\varphi}. N$ is assigned to the conclusion of $\to$Intro. The φ represents the assumption being discharged; only have we included it can we infer the formula of $\lambda x^{\varphi}. N$ based on the formula of N.

Definition 4.2 (Typing context)**.** A *typing context* is a mapping from variables to formulas. We will call it simply the "context" if there is no confusion. We write a context Γ as a set of pairs $\langle x, \varphi \rangle$.

A pair $\Gamma \Rightarrow M$ where M is a proof term represents a derivation of a formula with context Γ.

Definition 4.3 (Typing pair)**.** A *typing pair* is a pair $\langle \Gamma, M \rangle$, where Γ is a typing context and M is a proof term.

Since in general terms only make sense with specific contexts, we will speak simply of "terms" from now on instead of "typing pair"; and it will be apparent when we are talking about the literal term M.

4.4 Converting Derivations to Proof Terms

We will describe the process of converting natural deduction derivations to pairs. We will write a proof term to the left of each formula in the derivation, resulting in expressions of the form $M : \varphi$. We'll then say that, M *witnesses* φ. Let's call such an expression a *judgment*. int:pty:pt:sec

First let us assign to each assumption a variable, with the following constraints:

1. Assumptions discharged in the same step (that is, with the same number on the square bracket) must be assigned the same variable.
2. For assumptions not discharged, assumptions of different formulas should be assigned different variables.

Such an assignment translates all assumptions of the form

$$\varphi \qquad \text{into} \qquad x : \varphi.$$

With assumptions all associated with variables (which are terms), we can now inductively translate the rest of the deduction tree. The modified natural deduction rules taking into account context and proof terms are given below. Given the proof terms for the premise(s), we obtain the corresponding proof term for conclusion.

$$\frac{M_1 : \varphi_1 \qquad M_2 : \varphi_2}{\langle M_1, M_2 \rangle : \varphi_1 \wedge \varphi_2} \wedge\text{Intro}$$

$$\frac{M : \varphi_1 \wedge \varphi_2}{\mathrm{p}_i(M) : \varphi_1} \wedge\text{Elim}_1 \qquad \frac{M : \varphi_1 \wedge \varphi_2}{\mathrm{p}_i(M) : \varphi_2} \wedge\text{Elim}_2$$

In $\wedge$Intro we assume we have φ_1 witnessed by term M_1 and φ_2 witnessed by term M_2. We pack up the two terms into a pair $\langle M_1, M_2 \rangle$ which witnesses $\varphi_1 \wedge \varphi_2$.

In $\wedge\text{Elim}_i$ we assume that M witnesses $\varphi_1 \wedge \varphi_2$. The term witnessing φ_i is $\mathrm{p}_i(M)$. Note that M is not necessary of the form $\langle M_1, M_2\rangle$, so we cannot simply assign M_1 to the conclusion φ_i.

Note how this coincides with the BHK interpretation. What the BHK interpretation does not specify is how the function used as proof for $\varphi \to \psi$ is supposed to be obtained. If we think of proof terms as proofs or functions of proofs, we can be more explicit.

$$\dfrac{\begin{array}{c}[x : \varphi] \\ \vdots \\ N : \psi\end{array}}{\lambda x^{\varphi}.\, N : \varphi \to \psi}\;{\to}\text{Intro} \qquad \dfrac{P : \varphi \to \psi \qquad Q : \varphi}{PQ : \psi}\;{\to}\text{Elim}$$

The λ notation should be understood as the same as in the lambda calculus, and PQ means applying P to Q.

$$\dfrac{M_1 : \varphi_1}{\mathrm{in}_1^{\varphi_1}(M_1) : \varphi_1 \vee \varphi_2}\;\vee\text{Intro}_1 \qquad \dfrac{M_2 : \varphi_2}{\mathrm{in}_2^{\varphi_2}(M_2) : \varphi_1 \vee \varphi_2}\;\vee\text{Intro}_2$$

$$\dfrac{M : A_1 \vee \varphi_2 \qquad \begin{array}{c}[x_1 : \varphi_1] \\ \vdots \\ N_1 : \chi\end{array} \qquad \begin{array}{c}[x_2 : \varphi_2] \\ \vdots \\ N_2 : \chi\end{array}}{\mathrm{case}(M, x_1.N_1, x_2.N_2) : \chi}\;\vee\text{Elim}$$

The proof term $\mathrm{in}_1^{\varphi_1}(M_1)$ is a term witnessing $\varphi_1 \vee \varphi_2$, where M_1 witnesses φ_1.

The term $\mathrm{case}(M, x_1.N_1, x_2.N_2)$ mimics the case clause in programming languages: we already have the derivation of $\varphi \vee \psi$, a derivation of χ assuming φ, and a derivation of χ assuming ψ. The *case* operator thus select the appropriate proof depending on M; either way it's a proof of χ.

$$\dfrac{N : \bot}{\mathrm{contr}_{\varphi}(N) : \varphi}\;\bot_I$$

$\mathrm{contr}_{\varphi}(N)$ is a term witnessing φ, whenever N is a term witnessing $\bot$.

Now we have a natural deduction derivation with all formulas associated with a term. At each step, the relevant typing context Γ is given by the list of assumptions remaining undischarged at that step. Note that Γ is well defined: since we have forbidden assumptions of different undischarged assumptions to be assigned the same variable, there won't be any disagreement about the formulas mapped to which a variable is mapped.

We now give some examples of such translations:

Consider the derivation of $\neg\neg(\varphi \vee \neg\varphi)$, i.e., $((\varphi \vee (\varphi \to \bot)) \to \bot) \to \bot$. Its translation is:

$$
{}^{2}\dfrac{[y : (\varphi \lor (\varphi \to \bot)) \to \bot]^2 \qquad \dfrac{{}^{1}\dfrac{\dfrac{[y : (\varphi \lor (\varphi \to \bot)) \to \bot]^2 \qquad \dfrac{[x : \varphi]^1}{\mathrm{in}_1^{\varphi \to \bot}(x) : \varphi \lor (\varphi \to \bot)}}{y(\mathrm{in}_1^{\varphi \to \bot}(x)) : \bot}}{\lambda x^{\varphi}.\, y(\mathrm{in}_1^{\varphi \to \bot}(x)) : \varphi \to \bot}}{\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y(\mathrm{in}_1^{\varphi \to \bot}(x))) : \varphi \lor (\varphi \to \bot)}}{\dfrac{y(\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y\mathrm{in}_1^{\varphi \to \bot}(x))) : \bot}{\lambda y^{(\varphi \lor (\varphi \to \bot)) \to \bot}.\, y(\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y\mathrm{in}_1^{\varphi \to \bot}(x))) : ((\varphi \lor (\varphi \to \bot)) \to \bot) \to \bot}}
$$

The tree has no assumptions, so the context is empty; we get:

$$\vdash \lambda y^{(\varphi \lor (\varphi \to \bot)) \to \bot}.\, y(\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y\mathrm{in}_1^{\varphi \to \bot}(x))) : ((\varphi \lor (\varphi \to \bot)) \to \bot) \to \bot$$

If we leave out the last $\to$Intro, the assumptions denoted by y would be in the context and we would get:

$$y : ((\varphi \lor (\varphi \to \bot)) \to \bot) \vdash y(\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y\mathrm{in}_1^{\varphi \to \bot}(x))) : \bot$$

Another example: $\vdash \varphi \to (\varphi \to \bot) \to \bot$

$$
{}^{2}\dfrac{{}^{1}\dfrac{\dfrac{[x : \varphi]^2 \qquad [y : \varphi \to \bot]^1}{yx : \bot}}{\lambda y^{\varphi \to \bot}.\, yx : (\varphi \to \bot) \to \bot}}{\lambda x^{\varphi}.\, \lambda y^{\varphi \to \bot}.\, yx : \varphi \to (\varphi \to \bot) \to \bot}
$$

Again all assumptions are discharged and thus the context is empty, the resulting term is

$$\vdash \lambda x^{\varphi}.\, \lambda y^{\varphi \to \bot}.\, yx : \varphi \to (\varphi \to \bot) \to \bot$$

If we leave out the last two $\to$Intro inferences, the assumptions denoted by both x and y would be in context and we would get

$$x : \varphi, y : \varphi \to \bot \vdash yx : \bot$$

4.5 Recovering Derivations from Proof Terms

Now let us consider the other direction: translating terms back to natural deduction trees. We will use still use the double refutation of the excluded middle as example, and let S denote this term, i.e., int:pty:tp:sec

$$\lambda y^{(\varphi \lor (\varphi \to \bot)) \to \bot}.\, y(\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y\mathrm{in}_1^{\varphi \to \bot}(x))) : ((\varphi \lor (\varphi \to \bot)) \to \bot) \to \bot$$

For each natural deduction rule, the term in the conclusion is always formed by wrapping some operator around the terms assigned to the premise(s). Rules correspond uniquely to such operators. For example, from the structure of the S we infer that the last rule applied must be $\to$Intro, since it is of the form $\lambda y^{\cdots}. \ldots$, and the λ operator corresponds to $\to$Intro. In general we can recover the skeleton of the derivation solely by the structure of the term, e.g.,

$$
{}^{2}\dfrac{\dfrac{[y :]^2 \qquad \dfrac{{}^{1}\dfrac{\dfrac{[y :]^2 \qquad \dfrac{[x]^1}{\mathrm{in}_1^{\varphi\to\bot}(x) :}\,\lor\mathrm{Intro}_1}{y(\mathrm{in}_1^{\varphi\to\bot}(x)) :}\,\to\mathrm{Elim}}{\lambda x^{\varphi}.\, y(\mathrm{in}_1^{\varphi\to\bot}(x)) :}\,\to\mathrm{Intro}}{\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y\mathrm{in}_1^{\varphi\to\bot}(x)) :}\,\lor\mathrm{Intro}_2}{y(\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y\mathrm{in}_1^{\varphi\to\bot}(x))) :}\,\to\mathrm{Elim}}{\lambda y^{(\varphi\lor(\varphi\to\bot))\to\bot}.\, y(\mathrm{in}_2^{\varphi}(\lambda x^{\varphi}.\, y(\mathrm{in}_1^{\varphi\to\bot}(x)))) :}\,\to\mathrm{Intro}
$$

Our next step is to recover the formulas these terms witness. We define a function $F(\Gamma, M)$ which denotes the formula witnessed by M in context Γ, by induction on M as follows:

$$
\begin{aligned}
F(\Gamma, x) &= \Gamma(x) \\
F(\Gamma, \langle N_1, N_2\rangle) &= F(\Gamma, N_1) \land F(\Gamma, N_2) \\
F(\Gamma, \mathrm{p}_i(N)) &= \varphi_i \text{ if } F(\Gamma, N) = \varphi_1 \land \varphi_2 \\
F(\Gamma, \mathrm{in}_i^{\varphi}(N)) &= \begin{cases} F(N) \lor \varphi & \text{if } i = 1 \\ \varphi \lor F(N) & \text{if } i = 2 \end{cases} \\
F(\Gamma, \mathrm{case}(M, x_1.N_1, x_2.N_2)) &= F(\Gamma \cup \{x_i : F(\Gamma, M)\}, N_i) \\
F(\Gamma, \lambda x^{\varphi}.\, N) &= \varphi \to F(\Gamma \cup \{x : \varphi\}, N) \\
F(\Gamma, NM) &= \psi \text{ if } F(\Gamma, N) = \varphi \to \psi
\end{aligned}
$$

where $\Gamma(x)$ means the formula mapped to by x in Γ and $\Gamma \cup \{x : \varphi\}$ is a context exactly as Γ except mapping x to φ, whether or not x is already in Γ.

Note there are cases where $F(\Gamma, M)$ is not defined, for example:

1. In the first line, it is possible that x is not in Γ.
2. In recursive cases, the inner invocation may be undefined, making the outer one undefined too.
3. In the third line, its only defined when $F(\Gamma, M)$ is of the form $\varphi_1 \lor \varphi_2$, and the right hand is independent on i.

As we recursively compute $F(\Gamma, M)$, we work our way up the natural deduction derivation. The every step in the computation of $F(\Gamma, M)$ corresponds to a term in the derivation to which the derivation-to-term translation assigns M, and the formula computed is the end-formula of the derivation. However, the result may not be defined for some choices of Γ. We say that such pairs $\langle \Gamma, M\rangle$ are *ill-typed*, and otherwise *well-typed*. However, if the term M results from translating a derivation, and the formulas in Γ correspond to the undischarged assumptions of the derivation, the pair $\langle \Gamma, M\rangle$ will be well-typed.

Proposition 4.4. *If D is a derivation with undischarged assumptions $\varphi_1, \dots, \varphi_n$, M is the proof term associated with D and $\Gamma = \{x_1 : \varphi_1, \dots, x_n : \varphi_n\}$, then the result of recovering derivation from M in context Γ is D.*

In the other direction, if we first translate a typing pair to natural deduction and then translate it back, we won't get the same pair back since the choice of variables for the undischarged assumptions is underdetermined. For example, consider the pair $\langle \{x : \varphi, y : \varphi \to \psi\}, yx\rangle$. The corresponding derivation is

$$\dfrac{\varphi \to \psi \qquad \varphi}{\psi}\ \to\text{Elim}$$

By assigning different variables to the undischarged assumptions, say, u to $\varphi \to \psi$ and v to φ, we would get the term uv rather than yx. There is a connection, though: the terms will be the same up to renaming of variables.

Now we have established the correspondence between typing pairs and natural deduction, we can prove theorems for typing pairs and transfer the result to natural deduction derivations.

Similar to what we did in the natural deduction section, we can make some observations here too. Let $\Gamma \vdash M : \varphi$ denote that there is a pair (Γ, M) witnessing the formula φ. Then always $\Gamma \vdash x : \varphi$ if $x : \varphi \in \Gamma$, and the following rules are valid:

$$\dfrac{\Gamma \vdash M_1 : \varphi_1 \qquad \Delta \vdash M_2 : \varphi_2}{\Gamma, \Delta \vdash \langle M_1, M_2\rangle : \varphi_1 \land \varphi_2}\ \land\text{Intro} \qquad \dfrac{\Gamma \vdash M : \varphi_1 \land \varphi_2}{\Gamma \vdash \mathrm{p}_i(M) : \varphi_i}\ \land\text{Elim}_i$$

$$\dfrac{\Gamma \vdash M_1 : \varphi_1}{\Gamma \vdash \mathrm{in}_1^{\varphi_2}(M) : \varphi_1 \lor \varphi_2}\ \lor\text{Intro}_1 \qquad \dfrac{\Gamma \vdash M_2 : \varphi_2}{\Gamma \vdash \mathrm{in}_2^{\varphi_1}(M) : \varphi_1 \lor \varphi_2}\ \lor\text{Intro}_2$$

$$\dfrac{\Gamma \vdash M : \varphi \lor \psi \qquad \Delta_1, x_1 : \varphi_1 \vdash N_1 : \chi \qquad \Delta_2, x_2 : \varphi_2 \vdash N_2 : \chi}{\Gamma, \Delta, \Delta' \vdash \mathrm{case}(M, x_1.N_1, x_2.N_2) : \chi}\ \lor\text{Elim}$$

$$\dfrac{\Gamma, x : \varphi \vdash N : \psi}{\Gamma \vdash \lambda x^{\varphi}.\, N : \varphi \to \psi}\ \to\text{Intro} \qquad \dfrac{\Gamma \vdash Q : \varphi \qquad \Delta \vdash P : \varphi \to \psi}{\Gamma, \Delta \vdash PQ : \psi}\ \to\text{Elim}$$

$$\dfrac{\Gamma \vdash M : \bot}{\Gamma \vdash \mathrm{contr}_{\varphi}(M) : \varphi}\ \bot\text{Elim}$$

These are the typing rules of the simply typed lambda calculus extended with product, sum and bottom.

In addition, the $F(\Gamma, M)$ is actually a type checking algorithm; it returns the type of the term with respect to the context, or is undefined if the term is ill-typed with respect to the context.

4.6 Reduction

int:pty:red:sec

In natural deduction derivations, an introduction rule that is followed by an elimination rule is redundant. For instance, the derivation

$$\cfrac{\cfrac{\cfrac{\varphi \qquad \varphi \to \psi}{\psi}\;\to\text{Elim} \qquad [\chi]}{\cfrac{\psi \land \chi}{\psi}\;\land\text{Elim}}\;\land\text{Intro}}{\chi \to \psi}\;\to\text{Intro}$$

can be replaced with the simpler derivation:

$$\cfrac{\cfrac{\varphi \qquad \varphi \to \psi}{\psi}\;\to\text{Elim}}{\chi \to \psi}\;\to\text{Intro}$$

As we see, an $\land$Intro followed by $\land$Elim "cancel out." In general, we see that the conclusion of $\land$Elim is always the formula on one side of the conjunction, and the premises of $\land$Intro requires both sides of the conjunction, thus if we need a derivation of either side, we can simply use that derivation without introducing the conjunction followed by eliminating it.

Thus in general we have

$$\cfrac{\cfrac{\begin{matrix}\vdots\, D_1 \\ \varphi_1\end{matrix} \qquad \begin{matrix}\vdots\, D_2 \\ \varphi_2\end{matrix}}{\varphi_1 \land \varphi_2}\;\land\text{Intro}}{\varphi_i}\;\land\text{Elim}_i \qquad \rhd_1 \qquad \begin{matrix}\vdots\, D_i \\ \varphi_i\end{matrix}$$

The $\rhd_1$ symbol has a similar meaning as in the lambda calculus, i.e., a single step of a reduction. In the proof term syntax for derivations, the above reduction rule thus becomes:

$$(\Gamma, p_i\langle M_1^{\varphi_1}, M_2^{\varphi_2}\rangle) \rhd_1 (\Gamma, M_i)$$

In the typed lambda calculus, this is the beta reduction rule for the product type.

Note the type annotation on M_1 and M_2: while in the standard term syntax only $\lambda x^{\varphi}.\, N$ has such notion, we reuse the notation here to remind us of the formula the term is associated with in the corresponding natural deduction derivation, to reveal the correspondence between the two kinds of syntax.

In natural deduction, a pair of inferences such as those on the left, i.e., a pair that is subject to cancelling is called a *cut*. In the typed lambda calculus the term on the left of $\rhd_1$ is called a *redex*, and the term to the right is called the *reductum*. Unlike untyped lambda calculus, where only $(\lambda x.\, N)Q$ is considered to be redex, in the typed lambda calculus the syntax is extended to terms involving $\langle N, M\rangle$, $\mathrm{p}_i(N)$, $\mathrm{in}_i^{\varphi}(N)$, $\mathrm{case}(N, x_1.M_1, x_2.M_2)$, and $\mathrm{contr}_N()$, with corresponding redexes.

Similarly we have reduction for disjunction:

$$u\;\dfrac{\dfrac{\begin{matrix}\vdots\, D\\ \varphi_i\end{matrix}}{\varphi_1 \lor \varphi_2}\lor\text{Intro} \qquad \begin{matrix}[\varphi_1]^u\\ \vdots\, D_1\\ \chi\end{matrix} \qquad \begin{matrix}[\varphi_2]^u\\ \vdots\, D_2\\ \chi\end{matrix}}{\chi}\lor\text{Elim} \qquad \rhd_1 \qquad \begin{matrix}\vdots\, D\\ \varphi_i\\ \vdots\, D_i\\ \chi\end{matrix}$$

This corresponds to a reduction on proof terms:

$$(\Gamma, \mathrm{case}(\mathrm{in}_i^{\varphi_i}(M^{\varphi_i}), x_1^{\varphi_1}.N_1^{\chi}, x_2^{\varphi_2}.N_2^{\chi})) \rhd_1 (\Gamma, N_i^{\chi}[M^{\varphi_i}/x_i^{\varphi_i}])$$

This is the beta reduction rule of for sum types. Here, $M[N/x]$ means replacing all assumptions denoted by variable x in M with N,

It would be nice if we pass the context Γ to the substitution function so that it can check if the substitution makes sense. For example, $xy[ab/y]$ does not make sense under the context $\{x : \varphi \to \theta, y : \varphi, a : \psi \to \chi, b : \psi\}$ since then we would be substituting a derivation of χ where a derivation of φ is expected. However, as long as our usage of substitution is careful enough to avoid such errors, we won't have to worry about such conflicts. Thus we can define it recursively as we did for untyped lambda calculus as if we are dealing with untyped terms.

Finally, the reduction of the function type corresponds to removal of a detour of a $\to$Intro followd by a $\to$Elim.

$$\dfrac{u\;\dfrac{\begin{matrix}[\varphi]^u\\ \vdots\, D\\ \psi\end{matrix}}{\varphi \to \psi}\to\text{Intro} \qquad \begin{matrix}\vdots\, D'\\ \varphi\end{matrix}}{\psi}\to\text{Elim} \qquad \rhd_1 \qquad \begin{matrix}\vdots\, D'\\ \varphi\\ \vdots\, D\\ \psi\end{matrix}$$

For proof terms, this amounts to ordinary beta reduction:

$$(\Gamma, (\lambda x^{\varphi}.\, N^{\psi})Q^{\varphi}) \rhd_1 (\Gamma, N^{\psi}[Q^{\varphi}/x^{\varphi}])$$

Absurdity has only an elimination rule and no introduction rule, thus there is no such reduction for it.

Note that the above notion of reduction concerns only deductions with a cut at the end of a derivation. We would of course like to extend it to reduction of cuts anywhere in a derivation, or reductions of subterms of proof terms which constitute redexes. Note that, however, the conclusion of the reduction does not change after reduction, thus we are free to continue applying rules to both sides of $\triangleright_1$. The resulting pairs of trees constitutes an extended notion of reduction; it is analogous to compatibility in the untyped lambda calculus.

It's easy to see that the context Γ does not change during the reduction (both the original and the extended version), thus it's unnecessary to mention the context when we are discussing reductions. In what follows we will assume that every term is accompanied by a context which does no change during reduction. We then say "proof term" when we mean a proof term accompanied by a context which makes it well-typed.

As in lambda calculus, the notion of normal-form term and normal deduction is given:

Definition 4.5. A proof term with no redex is said to be in *normal form*; likewise, a derivation without cuts is a *normal derivation*. A proof term is in normal form if and only if its counterpart derivation is normal.

4.7 Normalization

int:pty:nor:sec In this section we prove that, via some reduction order, any deduction can be reduced to a normal deduction, which is called the *normalization property*. We will make use of the propositions-as-types correspondence: we show that every proof term can be reduced to a normal form; normalization for natural deduction derivations then follows.

Firstly we define some functions that measure the complexity of terms. The *length* $\mathrm{len}(\varphi)$ of a formulas is defined by

$$\begin{aligned}
\mathrm{len}(p) &= 0 \\
\mathrm{len}(\varphi \wedge \psi) &= \mathrm{len}(\varphi) + \mathrm{len}(\psi) + 1 \\
\mathrm{len}(\varphi \vee \psi) &= \mathrm{len}(\varphi) + \mathrm{len}(\psi) + 1 \\
\mathrm{len}(\varphi \to \psi) &= \mathrm{len}(\varphi) + \mathrm{len}(\psi) + 1.
\end{aligned}$$

The complexity of a redex M is measured by its *cut rank* $\mathrm{cr}(M)$:

$$\begin{aligned}
\mathrm{cr}((\lambda x^{\varphi}.\, N^{\psi})Q) &= \mathrm{len}(\varphi) + \mathrm{len}(\psi) + 1 \\
\mathrm{cr}(\mathrm{p}_i(\langle M^{\varphi}, N^{\psi}\rangle)) &= \mathrm{len}(\varphi) + \mathrm{len}(\psi) + 1 \\
\mathrm{cr}(\mathrm{case}(\mathrm{in}_i^{\varphi_i}(M^{\varphi_i}), x_1^{\varphi_1}.N_1^{\chi}, x_2^{\varphi_2}.N_2^{\chi})) &= \mathrm{len}(\varphi) + \mathrm{len}(\psi) + 1
\end{aligned}$$

The complexity of a proof term is measured by the most complex redex in it, and 0 if it is normal:

$$\mathrm{mr}(M) = \max\{\mathrm{cr}(N) | N \text{ is a sub term of } M \text{ and is redex}\}$$

int:pty:nor:lem:subst **Lemma 4.6.** *If $M[N^{\varphi}/x^{\varphi}]$ is a redex and $M \not\equiv x$, then one of the following cases holds:*

1. *M is itself a redex, or*
2. *M is of the form $\mathrm{p}_i(x)$, and N is of the form $\langle P_1, P_2\rangle$*

3. *M is of the form* $\text{case}(i, x_1.P_1, x_2.P_2)$*, and N is of the form* $\text{in}_i(Q)$

4. *M is of the form xQ, and N is of the form $\lambda x.\, P$*

In the first case, $\text{cr}(M[N/x]) = \text{cr}(M)$*; in the other cases,* $\text{cr}(M[N/x]) = \text{len}(\varphi))$.

Proof. Proof by induction on M.

1. If M is a single variable y and $y \not\equiv x$, then $y[N/x]$ is y, hence not a redex.

2. If M is of the form $\langle N_1, N_2 \rangle$, or $\lambda x.\, N$, or $\text{in}_i^\varphi(N)$, then $M[N^\varphi/x^\varphi]$ is also of that form, and so is not a redex.

3. If M is of the form $\text{p}_i(P)$, we consider two cases.

    a) If P is of the form $\langle P_1, P_2 \rangle$, then $M \equiv \text{p}_i(\langle P_1, P_2 \rangle)$ is a redex, and clearly
    $$M[N/x] \equiv \text{p}_i(\langle P_1[N/x], P_2[N/x] \rangle)$$
    is also a redex. The cut ranks are equal.

    b) If P is a single variable, it must be x to make the substitution a redex, and N must be of the form $\langle P_1, P_2 \rangle$. Now consider
    $$M[N/x] \equiv \text{p}_i(x)[\langle P_1, P_2 \rangle / x],$$
    which is $\text{p}_i(\langle P_1, P_2 \rangle)$. Its cut rank is equal to $\text{cr}(x)$, which is $\text{len}(\varphi)$.

The cases of $\text{case}(N, x_1.N_1, x_2.N_2)$ and PQ are similar. □

Lemma 4.7. *If M contracts to M', and* $\text{cr}(M) > \text{cr}(N)$ *for all proper redex sub-terms N of M, then* $\text{cr}(M) > \text{mr}(M')$.

Proof. Proof by cases.

1. If M is of the form $\text{p}_i(\langle M_1, M_2 \rangle)$, then M' is M_i; since any sub-term of M_i is also proper sub-term of M, the claim holds.

2. If M is of the form $(\lambda x^\varphi.\, N)Q^\varphi$, then M' is $N[Q^\varphi/x^\varphi]$. Consider a redex in M'. Either there is corresponding redex in N with equal cut rank, which is less than $\text{cr}(M)$ by assumption, or the cut rank equals $\text{len}(\varphi)$, which by definition is less than $\text{cr}((\lambda x^\varphi.\, N)Q)$.

3. If M is of the form
$$\text{case}(\text{in}_i(N^{\varphi_i}), x_1^{\varphi_1}.N_1^\chi, x_2^{\varphi_2}.N_2^\chi),$$
then $M' \equiv N_i[N/x_i^{\varphi_i}]$. Consider a redex in M'. Either there is corresponding redex in N_i with equal cut rank, which is less than $\text{cr}(M)$ by assumption; or the cut rank equals $\text{len}(\varphi_i)$, which by definition is less than $\text{cr}(\text{case}(\text{in}_i(N^{\varphi_i}), x_1^{\varphi_1}.N_1^\chi, x_2^{\varphi_2}.N_2^\chi))$.

□

Theorem 4.8. *All proof terms reduce to normal form; all derivations reduce to normal derivations.*

Proof. The second follows from the first. We prove the first by complete induction on $m = \mathrm{mr}(M)$, where M is a proof term.

1. If $m = 0$, M is already normal.
2. Otherwise, we proceed by induction on n, the number of redexes in M with cut rank equal to m.

   a) If $n = 1$, select any redex N such that $m = \mathrm{cr}(N) > \mathrm{cr}(P)$ for any proper sub-term P which is also a redex of course. Such a redex must exist, since any term only has finitely many subterms.

      Let N' denote the reductum of N. Now by the lemma $\mathrm{mr}(N') < \mathrm{mr}(N)$, thus we can see that n, the number of redexes with $\mathrm{cr}(=)m$ is decreased. So m is decreased (by 1 or more), and we can apply the inductive hypothesis for m.

   b) For the induction step, assume $n > 1$. the process is similar, except that n is only decreased to a positive number and thus m does not change. We simply apply the induction hypothesis for n.

□

The normalization of terms is actually not specific to the reduction order we chose. In fact, one can prove that regardless of the order in which redexes are reduced, the term always reduces to a normal form. This property is called *strong normalization*.

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Bibliography