

int.1 Natural Deduction

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sec

Natural deduction without the \perp_C rules is a standard **derivation** system for intuitionistic logic. We repeat the rules here and indicate the motivation using the BHK interpretation. In each case, we can think of a rule which allows us to conclude that if the premises have constructions, so does the conclusion.

Since natural deduction **derivations** have undischarged assumptions, we should consider such a **derivation**, say, of φ from **undischarged** assumptions Γ , as a function that turns constructions of all $\psi \in \Gamma$ into a construction of φ . If there is a **derivation** of φ from no **undischarged** assumptions, then there is a construction of φ in the sense of the BHK interpretation. For the purpose of the discussion, however, we'll suppress the Γ when not needed.

An assumption φ by itself is a **derivation** of φ from the **undischarged** assumption φ . This agrees with the BHK-interpretation: the identity function on constructions turns any construction of φ into a construction of φ .

Conjunction

$$\frac{\varphi_1 \quad \varphi_2}{\varphi_1 \wedge \varphi_2} \wedge\text{Intro} \qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_i} \wedge\text{Elim}_i \quad i \in \{1, 2\}$$

Suppose we have constructions N_1, N_2 of φ_1 and φ_2 , respectively. Then we also have a construction $\varphi_1 \wedge \varphi_2$, namely the pair $\langle N_1, N_2 \rangle$.

A construction of $\varphi_1 \wedge \varphi_2$ on the BHK interpretation is a pair $\langle N_1, N_2 \rangle$. So assume we have such a pair. Then we also have a construction of each conjunct: N_1 is a construction of φ_1 and N_2 is a construction of φ_2 .

Conditional

$$\frac{[\varphi]^u \quad \dots \quad \psi}{\varphi \rightarrow \psi} \rightarrow\text{Intro} \qquad \frac{\varphi \rightarrow \psi \quad \varphi}{\psi} \rightarrow\text{Elim}$$

If we have a **derivation** of ψ from **undischarged** assumption φ , then there is a function f that turns constructions of φ into constructions of ψ . That same function is a construction of $\varphi \rightarrow \psi$. So, if the premise of $\rightarrow\text{Intro}$ has a construction conditional on a construction of φ , the conclusion $\varphi \rightarrow \psi$ has a construction.

On the other hand, suppose there are constructions N of φ and f of $\varphi \rightarrow \psi$. A construction of $\varphi \rightarrow \psi$ is a function that turns constructions of φ into

constructions of ψ . So, $f(N)$ is a construction of ψ , i.e., the conclusion of \rightarrow Elim has a construction.

Disjunction

$$\frac{\varphi_i}{\varphi_1 \vee \varphi_2} \vee\text{Intro}_i \quad i \in \{1, 2\}$$

$$\frac{u \quad \frac{\varphi_1 \vee \varphi_2}{\chi} \quad \begin{array}{c} [\varphi_1]^u \\ \vdots \\ \chi \end{array} \quad \begin{array}{c} [\varphi_2]^u \\ \vdots \\ \chi \end{array}}{\chi} \vee\text{Elim}$$

If we have a construction N_i of φ_i we can turn it into a construction $\langle i, N_i \rangle$ of $\varphi_1 \vee \varphi_2$. On the other hand, suppose we have a construction of $\varphi_1 \vee \varphi_2$, i.e., a pair $\langle i, N_i \rangle$ where N_i is a construction of φ_i , and also functions f_1, f_2 , which turn constructions of φ_1, φ_2 , respectively, into constructions of χ . Then $f_i(N_i)$ is a construction of χ , the conclusion of \vee Elim.

Absurdity

$$\frac{\perp}{\varphi} \perp_I$$

If we have a **derivation** of \perp from **undischarged** assumptions ψ_1, \dots, ψ_n , then there is a function $f(M_1, \dots, M_n)$ that turns constructions of ψ_1, \dots, ψ_n into a construction of \perp . Since \perp has no construction, there cannot be any constructions of all of ψ_1, \dots, ψ_n either. Hence, f also has the property that *if* M_1, \dots, M_n are constructions of ψ_1, \dots, ψ_n , respectively, *then* $f(M_1, \dots, M_n)$ is a construction of φ .

Rules for \neg

Since $\neg\varphi$ is defined as $\varphi \rightarrow \perp$, we strictly speaking do not need rules for \neg . But if we did, this is what they'd look like:

$$\frac{n \quad \frac{\perp}{\neg\varphi} \neg\text{Intro} \quad \begin{array}{c} [\varphi]^n \\ \vdots \\ \perp \end{array}}{\neg\varphi} \neg\text{Intro}$$

$$\frac{\neg\varphi \quad \varphi}{\perp} \neg\text{Elim}$$

Examples of Derivations

1. $\vdash \varphi \rightarrow (\neg\varphi \rightarrow \perp)$, i.e., $\vdash \varphi \rightarrow ((\varphi \rightarrow \perp) \rightarrow \perp)$

$$\frac{\frac{\frac{[\varphi]^2}{\perp} \rightarrow\text{Elim}}{(\varphi \rightarrow \perp) \rightarrow \perp} \rightarrow\text{Intro}}{\varphi \rightarrow ((\varphi \rightarrow \perp) \rightarrow \perp)} \rightarrow\text{Intro}}{[\varphi \rightarrow \perp]^1}$$

2. $\vdash ((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$

$$\frac{\frac{\frac{\frac{[\varphi]^2}{\chi} \rightarrow\text{Intro}}{\psi \rightarrow \chi} \rightarrow\text{Intro}}{\varphi \rightarrow (\psi \rightarrow \chi)} \rightarrow\text{Intro}}{((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))} \rightarrow\text{Intro}}{\frac{\frac{[\varphi \wedge \psi \rightarrow \chi]^3}{\varphi \wedge \psi} \wedge\text{Elim}}{\frac{[\varphi]^2 \quad [\psi]^1}{\varphi \wedge \psi} \wedge\text{Intro}} \rightarrow\text{Elim}}{1}}$$

3. $\vdash \neg(\varphi \wedge \neg\varphi)$, i.e., $\vdash (\varphi \wedge (\varphi \rightarrow \perp)) \rightarrow \perp$

$$\frac{\frac{\frac{[\varphi \wedge (\varphi \rightarrow \perp)]^1}{\varphi \rightarrow \perp} \wedge\text{Elim}}{\perp} \rightarrow\text{Intro}}{\frac{[\varphi \wedge (\varphi \rightarrow \perp)]^1}{\varphi} \wedge\text{Elim}} \rightarrow\text{Elim}}{1}$$

4. $\vdash \neg\neg(\varphi \vee \neg\varphi)$, i.e., $\vdash ((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp) \rightarrow \perp$

$$\frac{\frac{\frac{\frac{[(\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp]^2}{\varphi \vee (\varphi \rightarrow \perp)} \vee\text{Intro}}{\perp} \rightarrow\text{Intro}}{((\varphi \vee (\varphi \rightarrow \perp)) \rightarrow \perp) \rightarrow \perp} \rightarrow\text{Intro}}{\frac{[\varphi]^1}{\varphi \vee (\varphi \rightarrow \perp)} \vee\text{Intro}} \rightarrow\text{Elim}}{2}$$

Proposition int.1. *If $\Gamma \vdash \varphi$ in intuitionistic logic, $\Gamma \vdash \varphi$ in classical logic. In particular, if φ is an intuitionistic theorem, it is also a classical theorem.*

Proof. Every natural deduction rule is also a rule in classical natural deduction, so every **derivation** in intuitionistic logic is also a **derivation** in classical logic. \square

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Bibliography