int.1 Natural Deduction

Natural deduction without the \(\bot_C\) rules is a standard derivation system for intuitionistic logic. We repeat the rules here and indicate the motivation using the BHK interpretation. In each case, we can think of a rule which allows us to conclude that if the premises have constructions, so does the conclusion.

Since natural deduction derivations have undischarged assumptions, we should consider such a derivation, say, of \(\varphi\) from undischarged assumptions \(\Gamma\), as a function that turns constructions of all \(\psi \in \Gamma\) into a construction of \(\varphi\). If there is a derivation of \(\varphi\) from no undischarged assumptions, then there is a construction of \(\varphi\) in the sense of the BHK interpretation. For the purpose of the discussion, however, we’ll suppress the \(\Gamma\) when not needed.

An assumption \(\varphi\) by itself is a derivation of \(\varphi\) from the undischarged assumption \(\varphi\). This agrees with the BHK-interpretation: the identity function on constructions turns any construction of \(\varphi\) into a construction of \(\varphi\).

Conjunction

\[
\frac{\varphi_1 \quad \varphi_2}{\varphi_1 \land \varphi_2} \land \text{Intro} \quad \quad \quad \frac{\varphi_1 \land \varphi_2}{\varphi_i} \land \text{Elim}_i \quad i \in \{1, 2\}
\]

Suppose we have constructions \(N_1, N_2\) of \(\varphi_1\) and \(\varphi_2\), respectively. Then we also have a construction \(\varphi_1 \land \varphi_2\), namely the pair \(\langle N_1, N_2 \rangle\).

A construction of \(\varphi_1 \land \varphi_2\) on the BHK interpretation is a pair \(\langle N_1, N_2 \rangle\). So assume we have such a pair. Then we also have a construction of each conjunct: \(N_1\) is a construction of \(\varphi_1\) and \(N_2\) is a construction of \(\varphi_2\).

Conditional

\[
\begin{array}{c}
\vdash \varphi \\
\vdash \psi \\
\vdash \varphi \rightarrow \psi \\
\varphi \rightarrow \psi \quad \rightarrow \text{Intro} \\
\varphi \rightarrow \psi \quad \psi \quad \rightarrow \text{Elim}
\end{array}
\]

If we have a derivation of \(\psi\) from undischarged assumption \(\varphi\), then there is a function \(f\) that turns constructions of \(\varphi\) into constructions of \(\psi\). That same function is a construction of \(\varphi \rightarrow \psi\). So, if the premise of \(\rightarrow \text{Intro}\) has a construction conditional on a construction of \(\varphi\), the conclusion \(\varphi \rightarrow \psi\) has a construction.

On the other hand, suppose there are constructions \(N\) of \(\varphi\) and \(f\) of \(\varphi \rightarrow \psi\). A construction of \(\varphi \rightarrow \psi\) is a function that turns constructions of \(\varphi\) into...
constructions of $\psi$. So, $f(N)$ is a construction of $\psi$, i.e., the conclusion of $\rightarrow$Elim has a construction.

**Disjunction**

\[
\begin{array}{c}
\frac{\varphi_i}{\varphi_1 \lor \varphi_2} \ \lor\text{Intro}_i \quad i \in \{1, 2\} \\
\frac{\varphi_1 \lor \varphi_2}{\chi \lor \chi} \ \lor\text{Elim}
\end{array}
\]

If we have a construction $N_i$ of $\varphi_i$ we can turn it into a construction $\langle i, N_i \rangle$ of $\varphi_1 \lor \varphi_2$. On the other hand, suppose we have a construction of $\varphi_1 \lor \varphi_2$, i.e., a pair $\langle i, N_i \rangle$ where $N_i$ is a construction of $\varphi_i$, and also functions $f_1, f_2$, which turn constructions of $\varphi_1, \varphi_2$, respectively, into constructions of $\chi$. Then $f_i(N_i)$ is a construction of $\chi$, the conclusion of $\lor$Elim.

**Absurdity**

\[
\frac{\bot}{\varphi} \ \bot\text{I}
\]

If we have a derivation of $\bot$ from undischarged assumptions $\psi_1, \ldots, \psi_n$, then there is a function $f(M_1, \ldots, M_n)$ that turns constructions of $\psi_1, \ldots, \psi_n$ into a construction of $\bot$. Since $\bot$ has no construction, there cannot be any constructions of all of $\psi_1, \ldots, \psi_n$ either. Hence, $f$ also has the property that if $M_1, \ldots, M_n$ are constructions of $\psi_1, \ldots, \psi_n$, respectively, then $f(M_1, \ldots, M_n)$ is a construction of $\varphi$.

**Rules for $\lnot$**

Since $\lnot \varphi$ is defined as $\varphi \rightarrow \bot$, we strictly speaking do not need rules for $\lnot$. But if we did, this is what they’d look like:

\[
\frac{[\varphi]^n}{\bot} \ \lnot\text{Intro}
\]

\[
\frac{\bot}{\varphi} \ \lnot\text{Elim}
\]
Examples of Derivations

1. \( \vdash \varphi \rightarrow (\neg \varphi \rightarrow \bot) \), i.e., \( \vdash (\varphi \rightarrow \bot) \rightarrow \bot \)

\[
\begin{array}{c}
[\varphi]^2 \\
[\varphi \rightarrow \bot]^1 \\
\downarrow \\
\bot \\
\hline
1 \\
(\varphi \rightarrow \bot) \rightarrow \bot \\
\hline
2 \\
\varphi \rightarrow (\varphi \rightarrow \bot) \rightarrow \bot \\
\hline
\end{array}
\rightarrow \text{Intro}
\]

\[
\begin{array}{c}
\vdash (\varphi \rightarrow \bot) \rightarrow \bot \\
\hline
\text{Intro}
\end{array}
\]

2. \( \vdash ((\varphi \land \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)) \)

\[
\begin{array}{c}
[\varphi \land \psi \rightarrow \chi]^3 \\
[\varphi]^2 \\
[\psi]^1 \\
\hline
\varphi \land \psi \\
\hline
\text{Intro}
\end{array}
\]

\[
\begin{array}{c}
\vdash (\varphi \land \psi) \rightarrow \chi \\
\hline
\vdash (\varphi \rightarrow (\psi \rightarrow \chi)) \\
\hline
\vdash (\varphi \land (\varphi \rightarrow \bot)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi)) \\
\hline
\vdash (\varphi \land \psi) \rightarrow \chi \\
\hline
\vdash (\varphi \rightarrow \bot) \rightarrow \bot \\
\hline
\text{Intro}
\end{array}
\]

3. \( \vdash (\neg \varphi \land \neg \varphi) \), i.e., \( \vdash (\varphi \land (\varphi \rightarrow \bot)) \rightarrow \bot \)

\[
\begin{array}{c}
[\varphi \land (\varphi \rightarrow \bot)]^1 \\
\text{Elim} \\
\hline
\varphi \rightarrow \bot \\
\hline
\text{Elim}
\end{array}
\]

\[
\begin{array}{c}
\vdash (\varphi \land (\varphi \rightarrow \bot)) \rightarrow \bot \\
\hline
\vdash (\varphi \land (\varphi \rightarrow \bot)) \rightarrow \bot \\
\hline
\vdash (\varphi \rightarrow \bot) \rightarrow \bot \\
\hline
\text{Intro}
\end{array}
\]

4. \( \vdash \neg \neg (\varphi \lor \neg \varphi) \), i.e., \( \vdash ((\varphi \lor (\varphi \rightarrow \bot)) \rightarrow \bot) \rightarrow \bot \)

\[
\begin{array}{c}
[\varphi \lor (\varphi \rightarrow \bot)]^2 \\
[\varphi]^1 \\
\hline
\varphi \lor (\varphi \rightarrow \bot) \\
\hline
\text{Intro}
\end{array}
\]

\[
\begin{array}{c}
\vdash (\varphi \lor (\varphi \rightarrow \bot)) \rightarrow \bot \\
\hline
\vdash (\varphi \rightarrow \bot) \rightarrow \bot \\
\hline
\vdash (\varphi \lor (\varphi \rightarrow \bot)) \rightarrow \bot \\
\hline
\end{array}
\rightarrow \text{Intro}
\]

\[
\begin{array}{c}
\vdash (\varphi \lor (\varphi \rightarrow \bot)) \rightarrow \bot \\
\hline
\vdash (\varphi \lor (\varphi \rightarrow \bot)) \rightarrow \bot \\
\hline
\vdash (\varphi \rightarrow \bot) \rightarrow \bot \\
\hline
\text{Intro}
\end{array}
\]

\[
\begin{array}{c}
\vdash (\varphi \lor (\varphi \rightarrow \bot)) \rightarrow \bot \\
\hline
\vdash (\varphi \lor (\varphi \rightarrow \bot)) \rightarrow \bot \\
\hline
\vdash (\varphi \rightarrow \bot) \rightarrow \bot \\
\hline
\text{Intro}
\end{array}
\]

Proposition int.1. If \( \Gamma \vdash \varphi \) in intuitionistic logic, \( \Gamma \vdash \varphi \) in classical logic. In particular, if \( \varphi \) is an intuitionistic theorem, it is also a classical theorem.

Proof. Every natural deduction rule is also a rule in classical natural deduction, so every derivation in intuitionistic logic is also a derivation in classical logic. \( \square \)