Chapter udf

Introduction

int.1 Constructive Reasoning

In constrast to extensions of classical logic by modal operators or second-order quantifiers, intuitionistic logic is "non-classical" in that it restricts classical logic. Classical logic is *non-constructive* in various ways. Intuitionistic logic is intended to capture a more "constructive" kind of reasoning characteristic of a kind of constructive mathematics. The following examples may serve to illustrate some of the underlying motivations.

Suppose someone claimed that they had determined a natural number n with the property that if n is even, the Riemann hypothesis is true, and if n is odd, the Riemann hypothesis is false. Great news! Whether the Riemann hypothesis is true or not is one of the big open questions of mathematics, and they seem to have reduced the problem to one of calculation, that is, to the determination of whether a specific number is prime or not.

What is the magic value of n? They describe it as follows: n is the natural number that is equal to 2 if the Riemann hypothesis is true, and 3 otherwise.

Angrily, you demand your money back. From a classical point of view, the description above does in fact determine a unique value of n; but what you really want is a value of n that is given *explicitly*.

To take another, perhaps less contrived example, consider the following question. We know that it is possible to raise an irrational number to a rational power, and get a rational result. For example, $\sqrt{2}^2 = 2$. What is less clear is whether or not it is possible to raise an irrational number to an *irrational* power, and get a rational result. The following theorem answers this in the affirmative:

Theorem int.1. *There are irrational numbers a and b such that a^b is rational.*

Proof. Consider $\sqrt{2}^{\sqrt{2}}$. If this is rational, we are done: we can let $a = b = \sqrt{2}$. Otherwise, it is irrational. Then we have

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2,$$

which is rational. So, in this case, let a be $\sqrt{2}^{\sqrt{2}}$, and let b be $\sqrt{2}$. □

Does this constitute a valid proof? Most mathematicians feel that it does. But again, there is something a little bit unsatisfying here: we have proved the existence of a pair of real numbers with a certain property, without being able to say *which* pair of numbers it is. It is possible to prove the same result, but in such a way that the pair a, b *is* given in the proof: take $a = \sqrt{3}$ and $b = \log_3 4$. Then

$$a^b = \sqrt{3}^{\log_3 4} = 3^{1/2 \cdot \log_3 4} = (3^{\log_3 4})^{1/2} = 4^{1/2} = 2,$$

since $3^{\log_3 x} = x$.

Intuitionistic logic is designed to capture a kind of reasoning where moves like the one in the first proof are disallowed. Proving the existence of an x satisfying $\varphi(x)$ means that you have to give a specific x, and a proof that it satisfies φ, like in the second proof. Proving that φ or ψ holds requires that you can prove one or the other.

Formally speaking, intuitionistic logic is what you get if you restrict a proof system for classical logic in a certain way. From the mathematical point of view, these are just formal deductive systems, but, as already noted, they are intended to capture a kind of mathematical reasoning. One can take this to be the kind of reasoning that is justified on a certain philosophical view of mathematics (such as Brouwer's intuitionism); one can take it to be a kind of mathematical reasoning which is more "concrete" and satisfying (along the lines of Bishop's constructivism); and one can argue about whether or not the formal description captures the informal motivation. But whatever philosophical positions we may hold, we can study intuitionistic logic as a formally presented logic; and for whatever reasons, many mathematical logicians find it interesting to do so.

int.2 Syntax of Intuitionistic Logic

The syntax of intuitionistic logic is the same as that for propositional logic. In classical propositional logic it is possible to define connectives by others, e.g., one can define $\varphi \rightarrow \psi$ by $\neg\varphi \lor \psi$, or $\varphi \lor \psi$ by $\neg(\neg\varphi \land \neg\psi)$. Thus, presentations of classical logic often introduce some connectives as abbreviations for these definitions. This is not so in intuitionistic logic, with two exceptions: $\neg\varphi$ can be—and often is—defined as an abbreviation for $\varphi \rightarrow \bot$. Then, of course, $\bot$ must not itself be defined! Also, $\varphi \leftrightarrow \psi$ can be defined, as in classical logic, as $(\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$. int:int:syn:sec

Formulas of propositional intuitionistic logic are built up from *propositional variables* and the propositional constant $\bot$ using *logical connectives*. We have:

1. A denumerable set At_0 of propositional variables p_0, p_1, ...

2. The propositional constant for falsity $\bot$.

3. The logical connectives: $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (conditional)

4. Punctuation marks: (,), and the comma.

int:int:syn:defn:formulas **Definition int.2** (Formula). The set $\mathrm{Frm}(\mathcal{L}_0)$ of *formulas* of propositional intuitionistic logic is defined inductively as follows:

1. $\bot$ is an atomic formula.

2. Every propositional variable p_i is an atomic formula.

3. If φ and ψ are formulas, then $(\varphi \wedge \psi)$ is a formula.

4. If φ and ψ are formulas, then $(\varphi \vee \psi)$ is a formula.

5. If φ and ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula.

6. Nothing else is a formula.

In addition to the primitive connectives introduced above, we also use the following *defined* symbols: $\neg$ (negation) and $\leftrightarrow$ (biconditional). Formulas constructed using the defined operators are to be understood as follows:

1. $\neg\varphi$ abbreviates $\varphi \rightarrow \bot$.

2. $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

Although $\neg$ is officially treated as an abbreviation, we will sometimes give explicit rules and clauses in definitions for $\neg$ as if it were primitive. This is mostly so we can state practice problems.

int.3 The Brouwer-Heyting-Kolmogorov Interpretation

int:int:bhk:sec

> Proofs of validity of intuitionistic propositions using the BHK interpretation are confusing; they have to be explained better.

There is an informal constructive interpretation of the intuitionist connectives, usually known as the Brouwer-Heyting-Kolmogorov interpretation. It uses the notion of a "construction," which you may think of as a constructive proof. (We don't use "proof" in the BHK interpretation so as not to get confused with the notion of a derivation in a formal proof system.) Based on this intuitive notion, the BHK interpretation explains the meanings of the intuitionistic connectives.

1. We assume that we know what constitutes a construction of an atomic statement.

2. A construction of $\varphi_1 \wedge \varphi_2$ is a pair $\langle M_1, M_2\rangle$ where M_1 is a construction of φ_1 and M_2 is a construction of A_2.
3. A construction of $\varphi_1 \vee \varphi_2$ is a pair $\langle s, M\rangle$ where s is 1 and M is a construction of φ_1, or s is 2 and M is a construction of φ_2.
4. A construction of $\varphi \to \psi$ is a function that converts a construction of φ into a construction of ψ.
5. There is no construction for $\perp$ (absurdity).
6. $\neg\varphi$ is defined as synonym for $\varphi \to \perp$. That is, a construction of $\neg\varphi$ is a function converting a construction of φ into a construction of $\perp$.

Example int.3. Take $\neg\perp$ for example. A construction of it is a function which, given any construction of $\perp$ as input, provides a construction of $\perp$ as output. Obviously, the identity function Id is such a construction: given a construction M of $\perp$, $\mathrm{Id}(M) = M$ yields a construction of $\perp$.

Generally speaking, $\neg\varphi$ means "A construction of φ is impossible".

Example int.4. Let us prove $\varphi \to \neg\neg\varphi$ for any proposition φ, which is $\varphi \to ((\varphi \to \perp) \to \perp)$. The construction should be a function f that, given a construction M of φ, returns a construction $f(M)$ of $(\varphi \to \perp) \to \perp$. Here is how f constructs the construction of $(\varphi \to \perp) \to \perp$: We have to define a function g which, when given a construction h of $\varphi \to \perp$ as input, outputs a construction of $\perp$. We can define g as follows: apply the input h to the construction M of φ (that we received earlier). Since the output $h(M)$ of h is a construction of $\perp$, $f(M)(h) = h(M)$ is a construction of $\perp$ if M is a construction of φ.

Example int.5. Let us give a construction for $\neg(\varphi \wedge \neg\varphi)$, i.e., $(\varphi \wedge (\varphi \to \perp)) \to \perp$. This is a function f which, given as input a construction M of $\varphi \wedge (\varphi \to \perp)$, yields a construction of $\perp$. A construction of a conjunction $\psi_1 \wedge \psi_2$ is a pair $\langle N_1, N_2\rangle$ where N_1 is a construction of ψ_1 and N_2 is a construction of ψ_2. We can define functions p_1 and p_2 which recover from a construction of $\psi_1 \wedge \psi_2$ the constructions of B_1 and ψ_2, respectively:

$$p_1(\langle N_1, N_2\rangle) = N_1$$
$$p_2(\langle N_1, N_2\rangle) = N_2$$

Here is what f does: First it applies p_1 to its input M. That yields a construction of φ. Then it applies p_2 to M, yielding a construction of $\varphi \to \perp$. Such a construction, in turn, is a function $p_2(M)$ which, if given as input a construction of φ, yields a construction of $\perp$. In other words, if we apply $p_2(M)$ to $p_1(M)$, we get a construction of $\perp$. Thus, we can define $f(M) = p_2(p_1(M))$.

Example int.6. Let us give a construction of $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$, i.e., a function f which turns a construction g of $(\varphi \wedge \psi) \to \chi$ into a construction of $(\varphi \to (\psi \to \chi))$. The construction g is itself a function (from constructions of $\varphi \wedge \psi$ to constructions of C). And the output $f(g)$ is a function h_g from constructions of φ to functions from constructions of ψ to constructions of χ.

Ok, this is confusing. We have to construct a certain function h_g, which will be the output of f for input g. The input of h_g is a construction M of φ. The output of $h_g(M)$ should be a function k_M from constructions N of ψ to constructions of χ. Let $k_{g,M}(N) = g(\langle M, N\rangle)$. Remember that $\langle M, N\rangle$ is a construction of $\varphi \wedge \psi$. So $k_{g,M}$ is a construction of $\psi \to \chi$: it maps constructions N of ψ to constructions of χ. Now let $h_g(M) = k_{g,M}$. That's a function that maps constructions M of φ to constructions $k_{g,M}$ of $\psi \to \chi$. Now let $f(g) = h_g$. That's a function that maps constructions g of $(\varphi \wedge \psi) \to \chi$ to constructions of $\varphi \to (\psi \to \chi)$. Whew!

The statement $\varphi \vee \neg\varphi$ is called the Law of Excluded Middle. We can prove it for some specific φ (e.g., $\perp \vee \neg\perp$), but not in general. This is because the intuitionistic disjunction requires a construction of one of the disjuncts, but there are statements which currently can neither be proved nor refuted (say, Goldbach's conjecture). However, you can't refute the law of excluded middle either: that is, $\neg\neg(\varphi \vee \neg\varphi)$ holds.

Example int.7. To prove $\neg\neg(\varphi \vee \neg\varphi)$, we need a function f that transforms a construction of $\neg(\varphi \vee \neg\varphi)$, i.e., of $(\varphi \vee (\varphi \to \perp)) \to \perp$, into a construction of $\perp$. In other words, we need a function f such that $f(g)$ is a construction of $\perp$ if g is a construction of $\neg(\varphi \vee \neg\varphi)$.

Suppose g is a construction of $\neg(\varphi \vee \neg\varphi)$, i.e., a function that transforms a construction of $\varphi \vee \neg\varphi$ into a construction of $\perp$. A construction of $\varphi \vee \neg\varphi$ is a pair $\langle s, M\rangle$ where either $s = 1$ and M is a construction of φ, or $s = 2$ and M is a construction of $\neg\varphi$. Let h_1 be the function mapping a construction M_1 of φ to a construction of $\varphi \vee \neg\varphi$: it maps M_1 to $\langle 1, M_2\rangle$. And let h_2 be the function mapping a construction M_2 of $\neg\varphi$ to a construction of $\varphi \vee \neg\varphi$: it maps M_2 to $\langle 2, M_2\rangle$.

Let k be $g \circ h_1$: it is a function which, if given a construction of φ, returns a construction of $\perp$, i.e., it is a construction of $\varphi \to \perp$ or $\neg\varphi$. Now let l be $g \circ h_2$. It is a function which, given a construction of $\neg\varphi$, provides a construction of $\perp$. Since k is a construction of $\neg\varphi$, $l(k)$ is a construction of $\perp$.

Together, what we've done is describe how we can turn a construction g of $\neg(\varphi \vee \neg\varphi)$ into a construction of $\perp$, i.e., the function f mapping a construction g of $\neg(\varphi \vee \neg\varphi)$ to the construction $l(k)$ of $\perp$ is a construction of $\neg\neg(\varphi \vee \neg\varphi)$.

As you can see, using the BHK interpretation to show the intuitionistic validity of formulas quickly becomes cumbersome and confusing. Luckily, there are better derivation systems for intuitionistic logic, and more precise semantic interpretations.

int.4 Natural Deduction

int:int:ntd:sec

Natural deduction without the $\bot_C$ rules is a standard derivation system for intuitionistic logic. We repeat the rules here and indicate the motivation using the BHK interpretation. In each case, we can think of a rule which allows us to conclude that if the premises have constructions, so does the conclusion.

Since natural deduction derivations have undischarged assumptions, we should consider such a derivation, say, of φ from undischarged assumptions Γ, as a function that turns constructions of all $\psi \in \Gamma$ into a construction of φ. If there is a derivation of φ from no undischarged assumptions, then there is a construction of φ in the sense of the BHK interpretation. For the purpose of the discussion, however, we'll suppress the Γ when not needed.

An assumption φ by itself is a derivation of φ from the undischarged assumption φ. This agrees with the BHK-interpretation: the identity function on constructions turns any construction of φ into a construction of φ.

Conjunction

$$\frac{\varphi_1 \qquad \varphi_2}{\varphi_1 \wedge \varphi_2}\wedge\text{Intro} \qquad\qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_i}\wedge\text{Elim}_i \quad i \in \{1, 2\}$$

Suppose we have constructions N_1, N_2 of φ_1 and φ_2, respectively. Then we also have a construction $\varphi_1 \wedge \varphi_2$, namely the pair $\langle N_1, N_2\rangle$.

A construction of $\varphi_1 \wedge \varphi_1$ on the BHK interpretation is a pair $\langle N_1, N_2\rangle$. So assume we have such a pair. Then we also have a construction of each conjunct: N_1 is a construction of φ_1 and N_2 is a construction of φ_2.

Conditional

$$\begin{array}{c} [\varphi]^u \\ \vdots \\ \psi \end{array}$$
$$\text{u}\ \frac{\psi}{\varphi \to \psi}\to\text{Intro} \qquad\qquad \frac{\varphi \to \psi \qquad \varphi}{\psi}\to\text{Elim}$$

If we have a derivation of ψ from undischarged assumption φ, then there is a function f that turns constructions of φ into constructions of ψ. That same function is a construction of $\varphi\to\psi$. So, if the premise of $\to$Intro has a construction conditional on a construction of φ, the conclusion $\varphi\to\psi$ has a construction.

On the other hand, suppose there are constructions N of φ and f of $\varphi \to \psi$. A construction of $\varphi \to \psi$ is a function that turns constructions of φ into constructions of ψ. So, $f(N)$ is a construction of ψ, i.e., the conclusion of $\to$Elim has a construction.

Disjunction

$$\frac{\varphi_i}{\varphi_1 \lor \varphi_2} \lor\text{Intro}_i \qquad i \in \{1, 2\}$$

$$u\,\frac{\varphi_1 \lor \varphi_2 \quad \begin{matrix}[\varphi_1]^u \\ \vdots \\ \chi\end{matrix} \quad \begin{matrix}[\varphi_2]^u \\ \vdots \\ \chi\end{matrix}}{\chi} \lor\text{Elim}$$

If we have a construction N_i of φ_i we can turn it into a construction $\langle i, N_i\rangle$ of $\varphi_1 \lor \varphi_2$. On the other hand, suppose we have a construction of $\varphi_1 \lor \varphi_2$, i.e., a pair $\langle i, N_i\rangle$ where N_i is a construction of φ_i, and also functions f_1, f_2, which turn constructions of φ_1, φ_2, respectively, into constructions of χ. Then $f_i(N_i)$ is a construction of χ, the conclusion of $\lor$Elim.

Absurdity

$$\frac{\bot}{\varphi} \bot_I$$

If we have a derivation of $\bot$ from undischarged assumptions $\psi_1, \ldots, \psi_n$, then there is a function $f(M_1, \ldots, M_n)$ that turns constructions of $\psi_1, \ldots, \psi_n$ into a construction of $\bot$. Since $\bot$ has no construction, there cannot be any constructions of all of $\psi_1, \ldots, \psi_n$ either. Hence, f also has the property that *if* $M_1, \ldots, M_n$ are constructions of $\psi_1, \ldots, \psi_n$, respectively, *then* $f(M_1, \ldots, M_n)$ is a construction of φ.

Rules for $\lnot$

Since $\lnot\varphi$ is defined as $\varphi \to \bot$, we strictly speaking do not need rules for $\lnot$. But if we did, this is what they'd look like:

$$n\,\frac{\begin{matrix}[\varphi]^n \\ \vdots \\ \bot\end{matrix}}{\lnot\varphi} \lnot\text{Intro} \qquad\qquad \frac{\lnot\varphi \quad \varphi}{\bot} \lnot\text{Elim}$$

Examples of Derivations

1. $\vdash \varphi \to (\lnot\varphi \to \bot)$, i.e., $\vdash \varphi \to ((\varphi \to \bot) \to \bot)$

$$
\begin{prooftree}
\AxiomC{$[\varphi]^2$}
\AxiomC{$[\varphi \to \bot]^1$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$(\varphi \to \bot) \to \bot$}
\LeftLabel{2}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi \to (\varphi \to \bot) \to \bot$}
\end{prooftree}
$$

2. $\vdash ((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$

$$
\begin{prooftree}
\AxiomC{$[(\varphi \wedge \psi) \to \chi]^3$}
\AxiomC{$[\varphi]^2$}
\AxiomC{$[\psi]^1$}
\RightLabel{$\wedge$Intro}
\BinaryInfC{$\varphi \wedge \psi$}
\RightLabel{$\to$Elim}
\BinaryInfC{χ}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$\psi \to \chi$}
\LeftLabel{2}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi \to (\psi \to \chi)$}
\LeftLabel{3}
\RightLabel{$\to$Intro}
\UnaryInfC{$((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$}
\end{prooftree}
$$

3. $\vdash \neg(\varphi \wedge \neg\varphi)$, i.e., $\vdash (\varphi \wedge (\varphi \to \bot)) \to \bot$

$$
\begin{prooftree}
\AxiomC{$[\varphi \wedge (\varphi \to \bot)]^1$}
\RightLabel{$\wedge$Elim}
\UnaryInfC{$\varphi \to \bot$}
\AxiomC{$[\varphi \wedge (\varphi \to \bot)]^1$}
\RightLabel{$\wedge$Elim}
\UnaryInfC{φ}
\RightLabel{$\to$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$(\varphi \wedge (\varphi \to \bot)) \to \bot$}
\end{prooftree}
$$

4. $\vdash \neg\neg(\varphi \vee \neg\varphi)$, i.e., $\vdash ((\varphi \vee (\varphi \to \bot)) \to \bot) \to \bot$

$$
\begin{prooftree}
\AxiomC{$[(\varphi \vee (\varphi \to \bot)) \to \bot]^2$}
\AxiomC{$[(\varphi \vee (\varphi \to \bot)) \to \bot]^2$}
\AxiomC{$[\varphi]^1$}
\RightLabel{$\vee$Intro}
\UnaryInfC{$\varphi \vee (\varphi \to \bot)$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{1}
\RightLabel{$\to$Intro}
\UnaryInfC{$\varphi \to \bot$}
\RightLabel{$\vee$Intro}
\UnaryInfC{$\varphi \vee (\varphi \to \bot)$}
\RightLabel{$\to$Elim}
\BinaryInfC{$\bot$}
\LeftLabel{2}
\RightLabel{$\to$Intro}
\UnaryInfC{$((\varphi \vee (\varphi \to \bot)) \to \bot) \to \bot$}
\end{prooftree}
$$

Proposition int.8. *If $\Gamma \vdash \varphi$ in intuitionistic logic, $\Gamma \vdash \varphi$ in classical logic. In particular, if φ is an intuitionistic theorem, it is also a classical theorem.*

Proof. Every natural deduction rule is also a rule in classical natural deduction, so every derivation in intuitionistic logic is also a derivation in classical logic. □

int.5 Axiomatic Derivations

int:int:axd:sec Axiomatic derivations for intuitionistic propositional logic are the conceptually simplest, and historically first, derivation systems. They work just as in classical propositional logic.

Definition int.9 (Derivability)**.** If Γ is a set of formulas of $\mathcal{L}$ then a *derivation* from Γ is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or

2. φ_i is an axiom; or

3. φ_i follows from some φ_j and φ_k with $j < i$ and $k < i$ by modus ponens, i.e., $\varphi_k \equiv \varphi_j \to \varphi_i$.

Definition int.10 (Axioms)**.** The set of Ax_0 of *axioms* for the intuitionistic propositional logic are all formulas of the following forms:

int:int:axd:ax:land1
$$(\varphi \land \psi) \to \varphi \tag{int.1}$$
int:int:axd:ax:land2
$$(\varphi \land \psi) \to \psi \tag{int.2}$$
int:int:axd:ax:land3
$$\varphi \to (\psi \to (\varphi \land \psi)) \tag{int.3}$$
int:int:axd:ax:lor1
$$\varphi \to (\varphi \lor \psi) \tag{int.4}$$
int:int:axd:ax:lor2
$$\varphi \to (\psi \lor \varphi) \tag{int.5}$$
int:int:axd:ax:lor3
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \lor \psi) \to \chi)) \tag{int.6}$$
int:int:axd:ax:lif1
$$\varphi \to (\psi \to \varphi) \tag{int.7}$$
int:int:axd:ax:lif2
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \tag{int.8}$$
int:int:axd:ax:lfalse1
$$\bot \to \varphi \tag{int.9}$$

Definition int.11 (Derivability)**.** A formula φ is *derivable* from Γ, written $\Gamma \vdash \varphi$, if there is a derivation from Γ ending in φ.

Definition int.12 (Theorems)**.** A formula φ is a *theorem* if there is a derivation of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Proposition int.13. *If $\Gamma \vdash \varphi$ in intuitionistic logic, $\Gamma \vdash \varphi$ in classical logic. In particular, if φ is an intuitionistic theorem, it is also a classical theorem.*

Proof. Every intuitionistic axiom is also a classical axiom, so every derivation in intuitionistic logic is also a derivation in classical logic. $\square$

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Bibliography