int.1 The Brouwer-Heyting-Kolmogorov Interpretation

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sec

> Proofs of validity of intuitionistic propositions using the BHK interpretation are confusing; they have to be explained better.

There is an informal constructive interpretation of the intuitionist connectives, usually known as the Brouwer-Heyting-Kolmogorov interpretation. It uses the notion of a "construction," which you may think of as a constructive proof. (We don't use "proof" in the BHK interpretation so as not to get confused with the notion of a derivation in a formal proof system.) Based on this intuitive notion, the BHK interpretation explains the meanings of the intuitionistic connectives.

1. We assume that we know what constitutes a construction of an atomic statement.
2. A construction of $\varphi_1 \wedge \varphi_2$ is a pair $\langle M_1, M_2 \rangle$ where M_1 is a construction of φ_1 and M_2 is a construction of A_2.
3. A construction of $\varphi_1 \vee \varphi_2$ is a pair $\langle s, M \rangle$ where s is 1 and M is a construction of φ_1, or s is 2 and M is a construction of φ_2.
4. A construction of $\varphi \rightarrow \psi$ is a function that converts a construction of φ into a construction of ψ.
5. There is no construction for $\perp$ (absurdity).
6. $\neg\varphi$ is defined as synonym for $\varphi \rightarrow \perp$. That is, a construction of $\neg\varphi$ is a function converting a construction of φ into a construction of $\perp$.

Example int.1. Take $\neg\perp$ for example. A construction of it is a function which, given any construction of $\perp$ as input, provides a construction of $\perp$ as output. Obviously, the identity function Id is such a construction: given a construction M of $\perp$, $\mathrm{Id}(M) = M$ yields a construction of $\perp$.

Generally speaking, $\neg\varphi$ means "A construction of φ is impossible".

Example int.2. Let us prove $\varphi \rightarrow \neg\neg\varphi$ for any proposition φ, which is $\varphi \rightarrow ((\varphi \rightarrow \perp) \rightarrow \perp)$. The construction should be a function f that, given a construction M of φ, returns a construction $f(M)$ of $(\varphi \rightarrow \perp) \rightarrow \perp$. Here is how f constructs the construction of $(\varphi \rightarrow \perp) \rightarrow \perp$: We have to define a function g which, when given a construction h of $\varphi \rightarrow \perp$ as input, outputs a construction of $\perp$. We can define g as follows: apply the input h to the construction M of φ (that we received earlier). Since the output $h(M)$ of h is a construction of $\perp$, $f(M)(h) = h(M)$ is a construction of $\perp$ if M is a construction of φ.

Example int.3. Let us give a construction for $\neg(\varphi \wedge \neg\varphi)$, i.e., $(\varphi \wedge (\varphi \to \bot)) \to \bot$. This is a function f which, given as input a construction M of $\varphi \wedge (\varphi \to \bot)$, yields a construction of $\bot$. A construction of a conjunction $\psi_1 \wedge \psi_2$ is a pair $\langle N_1, N_2 \rangle$ where N_1 is a construction of ψ_1 and N_2 is a construction of ψ_2. We can define functions p_1 and p_2 which recover from a construction of $\psi_1 \wedge \psi_2$ the constructions of B_1 and ψ_2, respectively:

$$p_1(\langle N_1, N_2 \rangle) = N_1$$
$$p_2(\langle N_1, N_2 \rangle) = N_2$$

Here is what f does: First it applies p_1 to its input M. That yields a construction of φ. Then it applies p_2 to M, yielding a construction of $\varphi \to \bot$. Such a construction, in turn, is a function $p_2(M)$ which, if given as input a construction of φ, yields a construction of $\bot$. In other words, if we apply $p_2(M)$ to $p_1(M)$, we get a construction of $\bot$. Thus, we can define $f(M) = p_2(p_1(M))$.

Example int.4. Let us give a construction of $((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$, i.e., a function f which turns a construction g of $(\varphi \wedge \psi) \to \chi$ into a construction of $(\varphi \to (\psi \to \chi))$. The construction g is itself a function (from constructions of $\varphi \wedge \psi$ to constructions of C). And the output $f(g)$ is a function h_g from constructions of φ to functions from constructions of ψ to constructions of χ.

Ok, this is confusing. We have to construct a certain function h_g, which will be the output of f for input g. The input of h_g is a construction M of φ. The output of $h_g(M)$ should be a function k_M from constructions N of ψ to constructions of χ. Let $k_{g,M}(N) = g(\langle M, N \rangle)$. Remember that $\langle M, N \rangle$ is a construction of $\varphi \wedge \psi$. So $k_{g,M}$ is a construction of $\psi \to \chi$: it maps constructions N of ψ to constructions of χ. Now let $h_g(M) = k_{g,M}$. That's a function that maps constructions M of φ to constructions $k_{g,M}$ of $\psi \to \chi$. Now let $f(g) = h_g$. That's a function that maps constructions g of $(\varphi \wedge \psi) \to \chi$ to constructions of $\varphi \to (\psi \to \chi)$. Whew!

The statement $\varphi \vee \neg\varphi$ is called the Law of Excluded Middle. We can prove it for some specific φ (e.g., $\bot \vee \neg\bot$), but not in general. This is because the intuitionistic disjunction requires a construction of one of the disjuncts, but there are statements which currently can neither be proved nor refuted (say, Goldbach's conjecture). However, you can't refute the law of excluded middle either: that is, $\neg\neg(\varphi \vee \neg\varphi)$ holds.

Example int.5. To prove $\neg\neg(\varphi \vee \neg\varphi)$, we need a function f that transforms a construction of $\neg(\varphi \vee \neg\varphi)$, i.e., of $(\varphi \vee (\varphi \to \bot)) \to \bot$, into a construction of $\bot$. In other words, we need a function f such that $f(g)$ is a construction of $\bot$ if g is a construction of $\neg(\varphi \vee \neg\varphi)$.

Suppose g is a construction of $\neg(\varphi \vee \neg\varphi)$, i.e., a function that transforms a construction of $\varphi \vee \neg\varphi$ into a construction of $\bot$. A construction of $\varphi \vee \neg\varphi$ is a pair $\langle s, M \rangle$ where either $s = 1$ and M is a construction of φ, or $s = 2$ and M is a construction of $\neg\varphi$. Let h_1 be the function mapping a construction M_1 of φ to a construction of $\varphi \vee \neg\varphi$: it maps M_1 to $\langle 1, M_2 \rangle$. And let h_2 be the function mapping a construction M_2 of $\neg\varphi$ to a construction of $\varphi \lor \neg\varphi$: it maps M_2 to $\langle 2, M_2 \rangle$.

Let k be $g \circ h_1$: it is a function which, if given a construction of φ, returns a construction of $\bot$, i.e., it is a construction of $\varphi \to \bot$ or $\neg\varphi$. Now let l be $g \circ h_2$. It is a function which, given a construction of $\neg\varphi$, provides a construction of $\bot$. Since k is a construction of $\neg\varphi$, $l(k)$ is a construction of $\bot$.

Together, what we've done is describe how we can turn a construction g of $\neg(\varphi \lor \neg\varphi)$ into a construction of $\bot$, i.e., the function f mapping a construction g of $\neg(\varphi \lor \neg\varphi)$ to the construction $l(k)$ of $\bot$ is a construction of $\neg\neg(\varphi \lor \neg\varphi)$.

As you can see, using the BHK interpretation to show the intuitionistic validity of formulas quickly becomes cumbersome and confusing. Luckily, there are better derivation systems for intuitionistic logic, and more precise semantic interpretations.

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Bibliography