int.1 Axiomatic Derivations

Axiomatic derivations for intuitionistic propositional logic are the conceptually simplest, and historically first, derivation systems. They work just as in classical propositional logic.

Definition int.1 (Derivability). If $\Gamma$ is a set of formulas of $L$ then a derivation from $\Gamma$ is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. $\varphi_i$ is an axiom; or
3. $\varphi_i$ follows from some $\varphi_j$ and $\varphi_k$ with $j < i$ and $k < i$ by modus ponens, i.e., $\varphi_k \equiv \varphi_j \rightarrow \varphi_i$.

Definition int.2 (Axioms). The set of $A_0$ of axioms for the intuitionistic propositional logic are all formulas of the following forms:

1. $(\varphi \land \psi) \rightarrow \varphi$  
2. $(\varphi \land \psi) \rightarrow \psi$  
3. $\varphi \rightarrow (\psi \rightarrow (\varphi \land \psi))$  
4. $\varphi \rightarrow (\varphi \lor \psi)$  
5. $\varphi \rightarrow (\psi \lor \varphi)$  
6. $(\varphi \rightarrow \psi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \lor \psi \rightarrow \chi))$  
7. $\varphi \rightarrow (\psi \lor \varphi)$  
8. $(\varphi \rightarrow (\psi \lor \varphi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$  
9. $\perp \rightarrow \varphi$

Definition int.3 (Derivability). A formula $\varphi$ is derivable from $\Gamma$, written $\Gamma \vdash \varphi$, if there is a derivation from $\Gamma$ ending in $\varphi$.

Definition int.4 (Theorems). A formula $\varphi$ is a theorem if there is a derivation of $\varphi$ from the empty set. We write $\vdash \varphi$ if $\varphi$ is a theorem and $\nvdash \varphi$ if it is not.

Proposition int.5. If $\Gamma \vdash \varphi$ in intuitionistic logic, $\Gamma \vdash \varphi$ in classical logic. In particular, if $\varphi$ is an intuitionistic theorem, it is also a classical theorem.

Proof. Every intuitionistic axiom is also a classical axiom, so every derivation in intuitionistic logic is also a derivation in classical logic.

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Bibliography