int.1 Axiomatic Derivations

int:int:axd:sec

Axiomatic derivations for intuitionistic propositional logic are the conceptually simplest, and historically first, derivation systems. They work just as in classical propositional logic.

Definition int.1 (Derivability)**.** If Γ is a set of formulas of $\mathcal{L}$ then a *derivation* from Γ is a finite sequence $\varphi_1, \ldots, \varphi_n$ of formulas where for each $i \leq n$ one of the following holds:

1. $\varphi_i \in \Gamma$; or
2. φ_i is an axiom; or
3. φ_i follows from some φ_j and φ_k with $j < i$ and $k < i$ by modus ponens, i.e., $\varphi_k \equiv \varphi_j \to \varphi_i$.

Definition int.2 (Axioms)**.** The set of Ax_0 of *axioms* for the intuitionistic propositional logic are all formulas of the following forms:

int:int:axd:ax:land1
$$(\varphi \land \psi) \to \varphi \tag{1}$$
int:int:axd:ax:land2
$$(\varphi \land \psi) \to \psi \tag{2}$$
int:int:axd:ax:land3
$$\varphi \to (\psi \to (\varphi \land \psi)) \tag{3}$$
int:int:axd:ax:lor1
$$\varphi \to (\varphi \lor \psi) \tag{4}$$
int:int:axd:ax:lor2
$$\varphi \to (\psi \lor \varphi) \tag{5}$$
int:int:axd:ax:lor3
$$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \lor \psi) \to \chi)) \tag{6}$$
int:int:axd:ax:lif1
$$\varphi \to (\psi \to \varphi) \tag{7}$$
int:int:axd:ax:lif2
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \tag{8}$$
int:int:axd:ax:lfalse1
$$\bot \to \varphi \tag{9}$$

Definition int.3 (Derivability)**.** A formula φ is *derivable* from Γ, written $\Gamma \vdash \varphi$, if there is a derivation from Γ ending in φ.

Definition int.4 (Theorems)**.** A formula φ is a *theorem* if there is a derivation of φ from the empty set. We write $\vdash \varphi$ if φ is a theorem and $\nvdash \varphi$ if it is not.

Proposition int.5. *If $\Gamma \vdash \varphi$ in intuitionistic logic, $\Gamma \vdash \varphi$ in classical logic. In particular, if φ is an intuitionistic theorem, it is also a classical theorem.*

Proof. Every intuitionistic axiom is also a classical axiom, so every derivation in intuitionistic logic is also a derivation in classical logic. □

Photo Credits

Bibliography