

## int.1 Axiomatic Derivations

int:int:axd:  
sec  
Axiomatic **derivations** for intuitionistic propositional logic are the conceptually simplest, and historically first, **derivation** systems. They work just as in classical propositional logic.

**Definition int.1 (Derivability).** If  $\Gamma$  is a set of **formulas** of  $\mathcal{L}$  then a **derivation** from  $\Gamma$  is a finite sequence  $\varphi_1, \dots, \varphi_n$  of **formulas** where for each  $i \leq n$  one of the following holds:

1.  $\varphi_i \in \Gamma$ ; or
2.  $\varphi_i$  is an axiom; or
3.  $\varphi_i$  follows from some  $\varphi_j$  and  $\varphi_k$  with  $j < i$  and  $k < i$  by modus ponens, i.e.,  $\varphi_k \equiv \varphi_j \rightarrow \varphi_i$ .

**Definition int.2 (Axioms).** The set of  $\text{Ax}_0$  of *axioms* for the intuitionistic propositional logic are all **formulas** of the following forms:

- |                          |   |     |
|--------------------------|---|-----|
| int:int:axd:             | $(\varphi \wedge \psi) \rightarrow \varphi$   | (1) |
| ax:land1<br>int:int:axd: | $(\varphi \wedge \psi) \rightarrow \psi$  | (2) |
| ax:land2<br>int:int:axd: | $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$  | (3) |
| ax:land3<br>int:int:axd: | $\varphi \rightarrow (\varphi \vee \psi)$   | (4) |
| ax:lor1<br>int:int:axd:  | $\varphi \rightarrow (\psi \vee \varphi)$   | (5) |
| ax:lor2<br>int:int:axd:  | $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$           | (6) |
| ax:lor3<br>int:int:axd:  | $\varphi \rightarrow (\psi \rightarrow \varphi)$  | (7) |
| ax:lif1<br>int:int:axd:  | $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$ | (8) |
| ax:lif2<br>int:int:axd:  | $\perp \rightarrow \varphi$   | (9) |

**Definition int.3 (Derivability).** A formula  $\varphi$  is **derivable** from  $\Gamma$ , written  $\Gamma \vdash \varphi$ , if there is a **derivation** from  $\Gamma$  ending in  $\varphi$ .

**Definition int.4 (Theorems).** A formula  $\varphi$  is a **theorem** if there is a **derivation** of  $\varphi$  from the empty set. We write  $\vdash \varphi$  if  $\varphi$  is a theorem and  $\not\vdash \varphi$  if it is not.

**Proposition int.5.** *If  $\Gamma \vdash \varphi$  in intuitionistic logic,  $\Gamma \vdash \varphi$  in classical logic. In particular, if  $\varphi$  is an intuitionistic theorem, it is also a classical theorem.*

*Proof.* Every intuitionistic axiom is also a classical axiom, so every **derivation** in intuitionistic logic is also a **derivation** in classical logic.  $\square$

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## Bibliography