Axiomatic Derivations

Axiomatic derivations for intuitionistic propositional logic are the conceptually simplest, and historically first, derivation systems. They work just as in classical propositional logic.

Definition int.1 (Derivability). If \( \Gamma \) is a set of formulas of \( L \) then a derivation from \( \Gamma \) is a finite sequence \( \varphi_1, \ldots, \varphi_n \) of formulas where for each \( i \leq n \) one of the following holds:

1. \( \varphi_i \in \Gamma \); or
2. \( \varphi_i \) is an axiom; or
3. \( \varphi_i \) follows from some \( \varphi_j \) and \( \varphi_k \) with \( j < i \) and \( k < i \) by modus ponens, i.e., \( \varphi_k \equiv \varphi_j \rightarrow \varphi_i \).

Definition int.2 (Axioms). The set of \( Ax_0 \) of axioms for the intuitionistic propositional logic are all formulas of the following forms:

\[(\varphi \land \psi) \rightarrow \varphi\] (1)
\[(\varphi \land \psi) \rightarrow \psi\] (2)
\[\varphi \rightarrow (\psi \rightarrow (\varphi \land \psi))\] (3)
\[\varphi \rightarrow (\varphi \lor \psi)\] (4)
\[\varphi \rightarrow (\psi \lor \varphi)\] (5)
\[((\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \lor \psi) \rightarrow \chi)))\] (6)
\[\varphi \rightarrow (\psi \rightarrow \varphi)\] (7)
\[((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\] (8)
\[\bot \rightarrow \varphi\] (9)

Definition int.3 (Derivability). A formula \( \varphi \) is derivable from \( \Gamma \), written \( \Gamma \vdash \varphi \), if there is a derivation from \( \Gamma \) ending in \( \varphi \).

Definition int.4 (Theorems). A formula \( \varphi \) is a theorem if there is a derivation of \( \varphi \) from the empty set. We write \( \vdash \varphi \) if \( \varphi \) is a theorem and \( \not\vdash \varphi \) if it is not.

Proposition int.5. If \( \Gamma \vdash \varphi \) in intuitionistic logic, \( \Gamma \vdash \varphi \) in classical logic. In particular, if \( \varphi \) is an intuitionistic theorem, it is also a classical theorem.

Proof. Every intuitionistic axiom is also a classical axiom, so every derivation in intuitionistic logic is also a derivation in classical logic.  

\( \square \)