### Axiomatic Derivations

Axiomatic derivations for intuitionistic propositional logic are the conceptually simplest, and historically first, derivation systems. They work just as in classical propositional logic.

**Definition int.1 (Derivability).** If \( \Gamma \) is a set of formulas of \( L \) then a derivation from \( \Gamma \) is a finite sequence \( \varphi_1, \ldots, \varphi_n \) of formulas where for each \( i \leq n \) one of the following holds:

1. \( \varphi_i \in \Gamma \); or
2. \( \varphi_i \) is an axiom; or
3. \( \varphi_i \) follows from some \( \varphi_j \) and \( \varphi_k \) with \( j < i \) and \( k < i \) by modus ponens, i.e., \( \varphi_k \equiv \varphi_j \rightarrow \varphi_i \).

**Definition int.2 (Axioms).** The set of \( Ax_0 \) of axioms for the intuitionistic propositional logic are all formulas of the following forms:

\[
\begin{align*}
(\varphi \land \psi) & \rightarrow \varphi \\
(\varphi \land \psi) & \rightarrow \psi \\
\varphi & \rightarrow (\psi \rightarrow (\varphi \land \psi)) \\
\varphi & \rightarrow (\varphi \lor \psi) \\
\varphi & \rightarrow (\psi \lor \varphi) \\
(\varphi \rightarrow \chi) & \rightarrow (((\psi \rightarrow \chi) \rightarrow ((\varphi \lor \psi) \rightarrow \chi))) \\
\varphi & \rightarrow (\psi \rightarrow \varphi) \\
(\varphi \rightarrow (\psi \rightarrow \chi)) & \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \\
\bot & \rightarrow \varphi
\end{align*}
\]

**Definition int.3 (Derivability).** A formula \( \varphi \) is derivable from \( \Gamma \), written \( \Gamma \vdash \varphi \), if there is a derivation from \( \Gamma \) ending in \( \varphi \).

**Definition int.4 (Theorems).** A formula \( \varphi \) is a theorem if there is a derivation of \( \varphi \) from the empty set. We write \( \vdash \varphi \) if \( \varphi \) is a theorem and \( \not\vdash \varphi \) if it is not.

**Proposition int.5.** If \( \Gamma \vdash \varphi \) in intuitionistic logic, \( \Gamma \vdash \varphi \) in classical logic. In particular, if \( \varphi \) is an intuitionistic theorem, it is also a classical theorem.

**Proof.** Every intuitionistic axiom is also a classical axiom, so every derivation in intuitionistic logic is also a derivation in classical logic. \( \square \)
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Bibliography