

tcp.1 \mathbf{Q} is C.e.-Complete

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Theorem tcp.1. \mathbf{Q} is c.e. but not decidable. In fact, it is a complete c.e. set.

Proof. It is not hard to see that \mathbf{Q} is c.e., since it is the set of (codes for) sentences y such that there is a proof x of y in \mathbf{Q} :

$$Q = \{y : \exists x \text{Prf}_{\mathbf{Q}}(x, y)\}.$$

But we know that $\text{Prf}_{\mathbf{Q}}(x, y)$ is computable (in fact, primitive recursive), and any set that can be written in the above form is c.e.

Saying that it is a complete c.e. set is equivalent to saying that $K \leq_m Q$, where $K = \{x : \varphi_x(x) \downarrow\}$. So let us show that K is reducible to \mathbf{Q} . Since Kleene's predicate $T(e, x, s)$ is primitive recursive, it is representable in \mathbf{Q} , say, by φ_T . Then for every x , we have

$$\begin{aligned} x \in K &\rightarrow \exists s T(x, x, s) \\ &\rightarrow \exists s (\mathbf{Q} \vdash \varphi_T(\bar{x}, \bar{x}, \bar{s})) \\ &\rightarrow \mathbf{Q} \vdash \exists s \varphi_T(\bar{x}, \bar{x}, s). \end{aligned}$$

Conversely, if $\mathbf{Q} \vdash \exists s \varphi_T(\bar{x}, \bar{x}, s)$, then, in fact, for some natural number n the formula $\varphi_T(\bar{x}, \bar{x}, \bar{n})$ must be true. Now, if $T(x, x, n)$ were false, \mathbf{Q} would prove $\neg \varphi_T(\bar{x}, \bar{x}, \bar{n})$, since φ_T represents T . But then \mathbf{Q} proves a false formula, which is a contradiction. So $T(x, x, n)$ must be true, which implies $\varphi_x(x) \downarrow$.

In short, we have that for every x , x is in K if and only if \mathbf{Q} proves $\exists s T(\bar{x}, \bar{x}, s)$. So the function f which takes x to (a code for) the sentence $\exists s T(\bar{x}, \bar{x}, s)$ is a reduction of K to \mathbf{Q} . \square

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Bibliography