

## tcp.1 Q is C.e.-Complete

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sec

**Theorem tcp.1.** *Q is c.e. but not decidable. In fact, it is a complete c.e. set.*

*Proof.* It is not hard to see that **Q** is c.e., since it is the set of (codes for) sentences  $y$  such that there is a proof  $x$  of  $y$  in **Q**:

$$Q = \{y : \exists x \text{Prf}_{\mathbf{Q}}(x, y)\}.$$

But we know that  $\text{Prf}_{\mathbf{Q}}(x, y)$  is computable (in fact, primitive recursive), and any set that can be written in the above form is c.e.

Saying that it is a complete c.e. set is equivalent to saying that  $K \leq_m Q$ , where  $K = \{x : \varphi_x(x) \downarrow\}$ . So let us show that  $K$  is reducible to **Q**. Since Kleene's predicate  $T(e, x, s)$  is primitive recursive, it is representable in **Q**, say, by  $\varphi_T$ . Then for every  $x$ , we have

$$\begin{aligned} x \in K &\rightarrow \exists s T(x, x, s) \\ &\rightarrow \exists s (\mathbf{Q} \vdash \varphi_T(\bar{x}, \bar{x}, \bar{s})) \\ &\rightarrow \mathbf{Q} \vdash \exists s \varphi_T(\bar{x}, \bar{x}, s). \end{aligned}$$

Conversely, if  $\mathbf{Q} \vdash \exists s \varphi_T(\bar{x}, \bar{x}, s)$ , then, in fact, for some natural number  $n$  the formula  $\varphi_T(\bar{x}, \bar{x}, \bar{n})$  must be true. Now, if  $T(x, x, n)$  were false, **Q** would prove  $\neg \varphi_T(\bar{x}, \bar{x}, \bar{n})$ , since  $\varphi_T$  represents  $T$ . But then **Q** proves a false formula, which is a contradiction. So  $T(x, x, n)$  must be true, which implies  $\varphi_x(x) \downarrow$ .

In short, we have that for every  $x$ ,  $x$  is in  $K$  if and only if **Q** proves  $\exists s T(\bar{x}, \bar{x}, s)$ . So the function  $f$  which takes  $x$  to (a code for) the sentence  $\exists s T(\bar{x}, \bar{x}, s)$  is a reduction of  $K$  to **Q**.  $\square$

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## Bibliography