tcp.1 Q is C.e.-Complete

inc:tcp:qce:sec

Theorem tcp.1. **Q** *is c.e. but not decidable. In fact, it is a complete c.e. set.*

Proof. It is not hard to see that **Q** is c.e., since it is the set of (codes for) sentences y such that there is a proof x of y in **Q**:

$$Q = \{y : \exists x \, \mathrm{Prf}_{\mathbf{Q}}(x, y)\}.$$

But we know that $\mathrm{Prf}_{\mathbf{Q}}(x, y)$ is computable (in fact, primitive recursive), and any set that can be written in the above form is c.e.

Saying that it is a complete c.e. set is equivalent to saying that $K \leq_m Q$, where $K = \{x : \varphi_x(x) \downarrow\}$. So let us show that K is reducible to **Q**. Since Kleene's predicate $T(e, x, s)$ is primitive recursive, it is representable in **Q**, say, by φ_T. Then for every x, we have

$$\begin{aligned} x \in K &\to \exists s \, T(x, x, s) \\ &\to \exists s \, (\mathbf{Q} \vdash \varphi_T(\overline{x}, \overline{x}, \overline{s})) \\ &\to \mathbf{Q} \vdash \exists s \, \varphi_T(\overline{x}, \overline{x}, s). \end{aligned}$$

Conversely, if $\mathbf{Q} \vdash \exists s \, \varphi_T(\overline{x}, \overline{x}, s)$, then, in fact, for some natural number n the formula $\varphi_T(\overline{x}, \overline{x}, \overline{n})$ must be true. Now, if $T(x, x, n)$ were false, **Q** would prove $\neg\varphi_T(\overline{x}, \overline{x}, \overline{n})$, since φ_T represents T. But then **Q** proves a false formula, which is a contradiction. So $T(x, x, n)$ must be true, which implies $\varphi_x(x) \downarrow$.

In short, we have that for every x, x is in K if and only if **Q** proves $\exists s \, T(\overline{x}, \overline{x}, s)$. So the function f which takes x to (a code for) the sentence $\exists s \, T(\overline{x}, \overline{x}, s)$ is a reduction of K to **Q**. □

Photo Credits

Bibliography