

## tcp.1 $\omega$ -Consistent Extensions of $\mathbf{Q}$ are Undecidable

inc:tcp:qgn:  
sec

The proof that  $\mathbf{Q}$  is c.e.-complete relied on the fact that any sentence provable in  $\mathbf{Q}$  is “true” of the natural numbers. The next definition and theorem strengthen this theorem, by pinpointing just those aspects of “truth” that were needed in the proof above. Don’t dwell on this theorem too long, though, because we will soon strengthen it even further. We include it mainly for historical purposes: Gödel’s original paper used the notion of  $\omega$ -consistency, but his result was strengthened by replacing  $\omega$ -consistency with ordinary consistency soon after.

explanation

inc:tcp:qgn:  
thm:oconsis-q

**Definition tcp.1.** A theory  $\mathbf{T}$  is  $\omega$ -consistent if the following holds: if  $\exists x \varphi(x)$  is any sentence and  $\mathbf{T}$  proves  $\neg\varphi(\bar{0}), \neg\varphi(\bar{1}), \neg\varphi(\bar{2}), \dots$  then  $\mathbf{T}$  does not prove  $\exists x \varphi(x)$ .

**Theorem tcp.2.** *Let  $\mathbf{T}$  be any  $\omega$ -consistent theory that includes  $\mathbf{Q}$ . Then  $\mathbf{T}$  is not decidable.*

*Proof.* If  $\mathbf{T}$  includes  $\mathbf{Q}$ , then  $\mathbf{T}$  represents the computable functions and relations. We need only modify the previous proof. As above, if  $x \in K$ , then  $\mathbf{T}$  proves  $\exists s \varphi_T(\bar{x}, \bar{x}, s)$ . Conversely, suppose  $\mathbf{T}$  proves  $\exists s \varphi_T(\bar{x}, \bar{x}, s)$ . Then  $x$  must be in  $K$ : otherwise, there is no halting computation of machine  $x$  on input  $x$ ; since  $\varphi_T$  represents Kleene’s  $T$  relation,  $\mathbf{T}$  proves  $\neg\varphi_T(\bar{x}, \bar{x}, \bar{0}), \neg\varphi_T(\bar{x}, \bar{x}, \bar{1}), \dots$ , making  $\mathbf{T}$   $\omega$ -inconsistent.  $\square$

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## Bibliography