

## tcp.1 Theories in which $\mathbf{Q}$ is Intepretable are Undecidable

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We can strengthen these results even more. Informally, an interpretation of a language  $\mathcal{L}_1$  in another language  $\mathcal{L}_2$  involves defining the universe, relation symbols, and function symbols of  $\mathcal{L}_1$  with **formulas** in  $\mathcal{L}_2$ . Though we won't take the time to do this, one can make this definition precise.

**Theorem tcp.1.** *Suppose  $\mathbf{T}$  is a theory in a language in which one can interpret the language of arithmetic, in such a way that  $\mathbf{T}$  is consistent with the interpretation of  $\mathbf{Q}$ . Then  $\mathbf{T}$  is undecidable. If  $\mathbf{T}$  proves the interpretation of the axioms of  $\mathbf{Q}$ , then no consistent extension of  $\mathbf{T}$  is decidable.*

The proof is just a small modification of the proof of the last theorem; one could use a counterexample to get a separation of  $\mathbf{Q}$  and  $\bar{\mathbf{Q}}$ . One can take **ZFC**, Zermelo-Fraenkel set theory with the axiom of choice, to be an axiomatic foundation that is powerful enough to carry out a good deal of ordinary mathematics. In **ZFC** one can define the natural numbers, and via this interpretation, the axioms of  $\mathbf{Q}$  are true. So we have

**Corollary tcp.2.** *There is no decidable extension of **ZFC**.*

**Corollary tcp.3.** *There is no complete, consistent, computably **axiomatizable** extension of **ZFC**.*

The language of **ZFC** has only a single binary relation,  $\in$ . (In fact, you don't even need equality.) So we have

**Corollary tcp.4.** *First-order logic for any language with a binary relation symbol is undecidable.*

This result extends to any language with two unary function symbols, since one can use these to simulate a binary relation symbol. The results just cited are tight: it turns out that first-order logic for a language with only *unary* relation symbols and at most one *unary* function symbol is decidable.

One more bit of trivia. We know that the set of sentences in the language  $0, ', +, \times, <$  true in the standard model is undecidable. In fact, one can define  $<$  in terms of the other symbols, and then one can define  $+$  in terms of  $\times$  and  $'$ . So the set of true sentences in the language  $0, ', \times$  is undecidable. On the other hand, Presburger has shown that the set of sentences in the language  $0, ', +$  true in the language of arithmetic is decidable. The procedure is computationally infeasible, however.

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## Bibliography