Theories in which \( Q \) is Interpretable are Undecidable

We can strengthen these results even more. Informally, an interpretation of a language \( \mathcal{L}_1 \) in another language \( \mathcal{L}_2 \) involves defining the universe, relation symbols, and function symbols of \( \mathcal{L}_1 \) with formulas in \( \mathcal{L}_2 \). Though we won’t take the time to do this, one can make this definition precise.

**Theorem tcp.1.** Suppose \( T \) is a theory in a language in which one can interpret the language of arithmetic, in such a way that \( T \) is consistent with the interpretation of \( Q \). Then \( T \) is undecidable. If \( T \) proves the interpretation of the axioms of \( Q \), then no consistent extension of \( T \) is decidable.

The proof is just a small modification of the proof of the last theorem; one could use a counterexample to get a separation of \( Q \) and \( \overline{Q} \). One can take ZFC, Zermelo-Fraenkel set theory with the axiom of choice, to be an axiomatic foundation that is powerful enough to carry out a good deal of ordinary mathematics. In ZFC one can define the natural numbers, and via this interpretation, the axioms of \( Q \) are true. So we have

**Corollary tcp.2.** There is no decidable extension of ZFC.

**Corollary tcp.3.** There is no complete, consistent, computably axiomatizable extension of ZFC.

The language of ZFC has only a single binary relation, \( \in \). (In fact, you don’t even need equality.) So we have

**Corollary tcp.4.** First-order logic for any language with a binary relation symbol is undecidable.

This result extends to any language with two unary function symbols, since one can use these to simulate a binary relation symbol. The results just cited are tight: it turns out that first-order logic for a language with only unary relation symbols and at most one unary function symbol is decidable.

One more bit of trivia. We know that the set of sentences in the language \( 0, , +, \times, < \) true in the standard model is undecidable. In fact, one can define \( < \) in terms of the other symbols, and then one can define \( + \) in terms of \( \times \) and \( ' \). So the set of true sentences in the language \( 0, , +, \times \) is undecidable. On the other hand, Presburger has shown that the set of sentences in the language \( 0, , + \) true in the language of arithmetic is decidable. The procedure is computationally infeasible, however.

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**Bibliography**