Theories in which Q is Interpretable are Undecidable

We can strengthen these results even more. Informally, an interpretation of a language $L_1$ in another language $L_2$ involves defining the universe, relation symbols, and function symbols of $L_1$ with formulas in $L_2$. Though we won’t take the time to do this, one can make this definition precise.

**Theorem tcp.1.** Suppose $T$ is a theory in a language in which one can interpret the language of arithmetic, in such a way that $T$ is consistent with the interpretation of Q. Then $T$ is undecidable. If $T$ proves the interpretation of the axioms of Q, then no consistent extension of $T$ is decidable.

The proof is just a small modification of the proof of the last theorem; one could use a counterexample to get a separation of Q and $\overline{Q}$. One can take ZFC, Zermelo-Fraenkel set theory with the axiom of choice, to be an axiomatic foundation that is powerful enough to carry out a good deal of ordinary mathematics. In ZFC one can define the natural numbers, and via this interpretation, the axioms of Q are true. So we have

**Corollary tcp.2.** There is no decidable extension of ZFC.

**Corollary tcp.3.** There is no complete, consistent, computably axiomatizable extension of ZFC.

The language of ZFC has only a single binary relation, $\in$. (In fact, you don’t even need equality.) So we have

**Corollary tcp.4.** First-order logic for any language with a binary relation symbol is undecidable.

This result extends to any language with two unary function symbols, since one can use these to simulate a binary relation symbol. The results just cited are tight: it turns out that first-order logic for a language with only unary relation symbols and at most one unary function symbol is decidable.

One more bit of trivia. We know that the set of sentences in the language $0, \prime, +, \times, <$ true in the standard model is undecidable. In fact, one can define $<$ in terms of the other symbols, and then one can define $+$ in terms of $\times$ and $\prime$. So the set of true sentences in the language $0, \prime, \times$ is undecidable. On the other hand, Presburger has shown that the set of sentences in the language $0, \prime, +$ true in the language of arithmetic is decidable. The procedure is computationally infeasible, however.

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Bibliography