Sentences Provable and Refutable in $Q$ are Computably Inseparable

Let $\overline{Q}$ be the set of sentences whose negations are provable in $Q$, i.e., $\overline{Q} = \{ \varphi : Q \vdash \neg \varphi \}$. Remember that disjoint sets $A$ and $B$ are said to be computably inseparable if there is no computable set $C$ such that $A \subseteq C$ and $B \subseteq \overline{C}$.

**Lemma tcp.1.** $Q$ and $\overline{Q}$ are computably inseparable.

**Proof.** Suppose $C$ is a computable set such that $Q \subseteq C$ and $\overline{Q} \subseteq \overline{C}$. Let $R(x, y)$ be the relation

$x$ codes a formula $\theta(u)$ and $\theta(\overline{u})$ is in $C$.

We will show that $R(x, y)$ is a universal computable relation, yielding a contradiction.

Suppose $S(y)$ is computable, represented by $\theta_S(u)$ in $Q$. Then

$$
S(\overline{u}) \rightarrow Q \vdash \theta_S(\overline{u}) \\
\rightarrow \theta_S(\overline{u}) \in C
$$

and

$$
\neg S(\overline{u}) \rightarrow Q \vdash \neg \theta_S(\overline{u}) \\
\rightarrow \theta_S(\overline{u}) \in \overline{Q} \\
\rightarrow \theta_S(\overline{u}) \notin C
$$

So $S(y)$ is equivalent to $R(\#(\theta_S(\overline{u})), y)$. $\square$

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**Bibliography**