

## tcp.1 Sentences Provable and Refutable in $\mathbf{Q}$ are Computably Inseparable

inc:tcp:ins:  
sec Let  $\bar{\mathbf{Q}}$  be the set of sentences whose *negations* are provable in  $\mathbf{Q}$ , i.e.,  $\bar{\mathbf{Q}} = \{\varphi : \mathbf{Q} \vdash \neg\varphi\}$ . Remember that disjoint sets  $A$  and  $B$  are said to be computably inseparable if there is no computable set  $C$  such that  $A \subseteq C$  and  $B \subseteq \bar{C}$ .

**Lemma tcp.1.**  $\mathbf{Q}$  and  $\bar{\mathbf{Q}}$  are computably inseparable.

*Proof.* Suppose  $C$  is a computable set such that  $\mathbf{Q} \subseteq C$  and  $\bar{\mathbf{Q}} \subseteq \bar{C}$ . Let  $R(x, y)$  be the relation

$x$  codes a formula  $\theta(u)$  and  $\theta(\bar{y})$  is in  $C$ .

We will show that  $R(x, y)$  is a universal computable relation, yielding a contradiction.

Suppose  $S(y)$  is computable, represented by  $\theta_S(u)$  in  $\mathbf{Q}$ . Then

$$\begin{aligned} S(\bar{n}) &\rightarrow \mathbf{Q} \vdash \theta_S(\bar{n}) \\ &\rightarrow \theta_S(\bar{n}) \in C \end{aligned}$$

and

$$\begin{aligned} \neg S(\bar{n}) &\rightarrow \mathbf{Q} \vdash \neg\theta_S(\bar{n}) \\ &\rightarrow \theta_S(\bar{n}) \in \bar{\mathbf{Q}} \\ &\rightarrow \theta_S(\bar{n}) \notin C \end{aligned}$$

So  $S(y)$  is equivalent to  $R(\#(\theta_S(\bar{u})), y)$ . □

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### Bibliography