

tcp.1 Sentences Provable and Refutable in \mathbf{Q} are Computably Inseparable

inc:tcp:ins:
sec Let $\bar{\mathbf{Q}}$ be the set of sentences whose *negations* are provable in \mathbf{Q} , i.e., $\bar{\mathbf{Q}} = \{\varphi : \mathbf{Q} \vdash \neg\varphi\}$. Remember that disjoint sets A and B are said to be computably inseparable if there is no computable set C such that $A \subseteq C$ and $B \subseteq \bar{C}$.

Lemma tcp.1. \mathbf{Q} and $\bar{\mathbf{Q}}$ are computably inseparable.

Proof. Suppose C is a computable set such that $\mathbf{Q} \subseteq C$ and $\bar{\mathbf{Q}} \subseteq \bar{C}$. Let $R(x, y)$ be the relation

x codes a formula $\theta(u)$ and $\theta(\bar{y})$ is in C .

We will show that $R(x, y)$ is a universal computable relation, yielding a contradiction.

Suppose $S(y)$ is computable, represented by $\theta_S(u)$ in \mathbf{Q} . Then

$$\begin{aligned} S(\bar{n}) &\rightarrow \mathbf{Q} \vdash \theta_S(\bar{n}) \\ &\rightarrow \theta_S(\bar{n}) \in C \end{aligned}$$

and

$$\begin{aligned} \neg S(\bar{n}) &\rightarrow \mathbf{Q} \vdash \neg\theta_S(\bar{n}) \\ &\rightarrow \theta_S(\bar{n}) \in \bar{\mathbf{Q}} \\ &\rightarrow \theta_S(\bar{n}) \notin C \end{aligned}$$

So $S(y)$ is equivalent to $R(\#(\theta_S(\bar{u})), y)$. □

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Bibliography