Sentences Provable and Refutable in $\mathcal{Q}$ are Computably Inseparable

Let $\mathcal{Q}$ be the set of sentences whose negations are provable in $\mathcal{Q}$, i.e., $\mathcal{Q} = \{ \varphi : \mathcal{Q} \vdash \neg \varphi \}$. Remember that disjoint sets $A$ and $B$ are said to be computably inseparable if there is no computable set $C$ such that $A \subseteq C$ and $B \subseteq \overline{C}$.

**Lemma tcp.1.** $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are computably inseparable.

**Proof.** Suppose $C$ is a computable set such that $\mathcal{Q} \subseteq C$ and $\overline{\mathcal{Q}} \subseteq \overline{C}$. Let $R(x, y)$ be the relation

\[ x \text{ codes a formula } \theta(u) \text{ and } \theta(\overline{y}) \text{ is in } C. \]

We will show that $R(x, y)$ is a universal computable relation, yielding a contradiction.

Suppose $S(y)$ is computable, represented by $\theta_S(u)$ in $\mathcal{Q}$. Then

\[
S(\overline{\pi}) \rightarrow \mathcal{Q} \vdash \theta_S(\overline{\pi}) \rightarrow \theta_S(\overline{\pi}) \in C
\]

and

\[
\neg S(\overline{\pi}) \rightarrow \mathcal{Q} \vdash \neg \theta_S(\overline{\pi}) \rightarrow \theta_S(\overline{\pi}) \in \overline{\mathcal{Q}} \rightarrow \theta_S(\overline{\pi}) \notin C
\]

So $S(y)$ is equivalent to $R(\#(\theta_S(\overline{\pi})), y)$. \hfill $\Box$

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**Bibliography**