Remember that a theory is consistent if it does not prove both \( \varphi \) and \( \neg \varphi \) for any formula \( \varphi \). Since anything follows from a contradiction, an inconsistent theory is trivial: every sentence is provable. Clearly, if a theory is \( \omega \)-consistent, then it is consistent. But being consistent is a weaker requirement (i.e., there are theories that are consistent but not \( \omega \)-consistent.). We can weaken the assumption in ?? to simple consistency to obtain a stronger theorem.

**Lemma tcp.1.** There is no “universal computable relation.” That is, there is no binary computable relation \( R(x, y) \), with the following property: whenever \( S(y) \) is a unary computable relation, there is some \( k \) such that for every \( y \), \( S(y) \) is true if and only if \( R(k, y) \) is true.

*Proof.* Suppose \( R(x, y) \) is a universal computable relation. Let \( S(y) \) be the relation \( \neg R(y, y) \). Since \( S(y) \) is computable, for some \( k \), \( S(y) \) is equivalent to \( R(k, y) \). But then we have that \( S(k) \) is equivalent to both \( R(k, k) \) and \( \neg R(k, k) \), which is a contradiction. \( \square \)

**Theorem tcp.2.** Let \( T \) be any consistent theory that includes \( Q \). Then \( T \) is not decidable.

*Proof.* Suppose \( T \) is a consistent, decidable extension of \( Q \). We will obtain a contradiction by using \( T \) to define a universal computable relation.

Let \( R(x, y) \) hold if and only if
\[
\begin{align*}
x & \text{ codes a formula } \theta(u), \text{ and } T \text{ proves } \theta(y).
\end{align*}
\]
Since we are assuming that \( T \) is decidable, \( R \) is computable. Let us show that \( R \) is universal. If \( S(y) \) is any computable relation, then it is representable in \( Q \) (and hence \( T \)) by a formula \( \theta_S(u) \). Then for every \( n \), we have
\[
S(\pi) \quad \rightarrow \quad T \vdash \theta_S(\pi)
\]
\[
\rightarrow \quad R(\#\theta_S(u)^\#, n)
\]
and
\[
\neg S(\pi) \quad \rightarrow \quad T \vdash \neg \theta_S(\pi)
\]
\[
\rightarrow \quad T \not\vdash \theta_S(\pi) \quad \text{(since } T \text{ is consistent)}
\]
\[
\rightarrow \quad \neg R(\#\theta_S(u)^\#, n).
\]
That is, for every \( y \), \( S(y) \) is true if and only if \( R(\#\theta_S(u)^\#, y) \) is. So \( R \) is universal, and we have the contradiction we were looking for. \( \square \)

Let “true arithmetic” be the theory \( \{ \varphi : N \models \varphi \} \), that is, the set of sentences in the language of arithmetic that are true in the standard interpretation.

**Corollary tcp.3.** True arithmetic is not decidable.