

tcp.1 Consistent Extensions of \mathbf{Q} are Undecidable

Remember that a theory is *consistent* if it does not prove both φ and $\neg\varphi$ for any formula φ . Since anything follows from a contradiction, an inconsistent theory is trivial: every sentence is provable. Clearly, if a theory is ω -consistent, then it is consistent. But being consistent is a weaker requirement (i.e., there are theories that are consistent but not ω -consistent.). We can weaken the assumption in ?? to simple consistency to obtain a stronger theorem.

Lemma tcp.1. *There is no “universal computable relation.” That is, there is no binary computable relation $R(x, y)$, with the following property: whenever $S(y)$ is a unary computable relation, there is some k such that for every y , $S(y)$ is true if and only if $R(k, y)$ is true.*

Proof. Suppose $R(x, y)$ is a universal computable relation. Let $S(y)$ be the relation $\neg R(y, y)$. Since $S(y)$ is computable, for some k , $S(y)$ is equivalent to $R(k, y)$. But then we have that $S(k)$ is equivalent to both $R(k, k)$ and $\neg R(k, k)$, which is a contradiction. \square

Theorem tcp.2. *Let \mathbf{T} be any consistent theory that includes \mathbf{Q} . Then \mathbf{T} is not decidable.*

Proof. Suppose \mathbf{T} is a consistent, decidable extension of \mathbf{Q} . We will obtain a contradiction by using \mathbf{T} to define a universal computable relation.

Let $R(x, y)$ hold if and only if

x codes a formula $\theta(u)$, and \mathbf{T} proves $\theta(\bar{y})$.

Since we are assuming that \mathbf{T} is decidable, R is computable. Let us show that R is universal. If $S(y)$ is any computable relation, then it is representable in \mathbf{Q} (and hence \mathbf{T}) by a formula $\theta_S(u)$. Then for every n , we have

$$\begin{aligned} S(\bar{n}) &\rightarrow T \vdash \theta_S(\bar{n}) \\ &\rightarrow R(\# \theta_S(u)^\#, n) \end{aligned}$$

and

$$\begin{aligned} \neg S(\bar{n}) &\rightarrow T \vdash \neg \theta_S(\bar{n}) \\ &\rightarrow T \not\vdash \theta_S(\bar{n}) \quad (\text{since } \mathbf{T} \text{ is consistent}) \\ &\rightarrow \neg R(\# \theta_S(u)^\#, n). \end{aligned}$$

That is, for every y , $S(y)$ is true if and only if $R(\# \theta_S(u)^\#, y)$ is. So R is universal, and we have the contradiction we were looking for. \square

Let “true arithmetic” be the theory $\{\varphi : \mathbb{N} \models \varphi\}$, that is, the set of sentences in the language of arithmetic that are true in the standard interpretation.

Corollary tcp.3. *True arithmetic is not decidable.*

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Bibliography