tcp.1  Consistent Extensions of Q are Undecidable

Remember that a theory is consistent if it does not prove both \( \varphi \) and \( \neg \varphi \) for any formula \( \varphi \). Since anything follows from a contradiction, an inconsistent theory is trivial: every sentence is provable. Clearly, if a theory is \( \omega \)-consistent, then it is consistent. But being consistent is a weaker requirement (i.e., there are theories that are consistent but not \( \omega \)-consistent.). We can weaken the assumption in ?? to simple consistency to obtain a stronger theorem.

Lemma tcp.1. There is no “universal computable relation.” That is, there is no binary computable relation \( R(x,y) \), with the following property: whenever \( S(y) \) is a unary computable relation, there is some \( k \) such that for every \( y \), \( S(y) \) is true if and only if \( R(k,y) \) is true.

Proof. Suppose \( R(x,y) \) is a universal computable relation. Let \( S(y) \) be the relation \( \neg R(y,y) \). Since \( S(y) \) is computable, for some \( k \), \( S(y) \) is equivalent to \( R(k,y) \). But then we have that \( S(k) \) is equivalent to both \( R(k,k) \) and \( \neg R(k,k) \), which is a contradiction.

Theorem tcp.2. Let \( T \) be any consistent theory that includes \( Q \). Then \( T \) is not decidable.

Proof. Suppose \( T \) is a consistent, decidable extension of \( Q \). We will obtain a contradiction by using \( T \) to define a universal computable relation.

Let \( R(x,y) \) hold if and only if

\[
\text{\( x \) codes a formula }  \theta(u), \text{ and }  \text{\( T \) proves } \theta(\overline{u}).
\]

Since we are assuming that \( T \) is decidable, \( R \) is computable. Let us show that \( R \) is universal. If \( S(y) \) is any computable relation, then it is representable in \( Q \) (and hence \( T \)) by a formula \( \theta_S(u) \). Then for every \( n \), we have

\[
S(\overline{n}) \rightarrow T \vdash \theta_S(\overline{n}) \rightarrow R(\overline{\theta_S(u)}^*, n)
\]

and

\[
\neg S(\overline{n}) \rightarrow T \vdash \neg \theta_S(\overline{n}) \rightarrow T \not\vdash \theta_S(\overline{n}) \quad \text{(since } T \text{ is consistent)} \rightarrow \neg R(\overline{\theta_S(u)}^*, n).
\]

That is, for every \( y \), \( S(y) \) is true if and only if \( R(\overline{\theta_S(u)}^*, y) \) is. So \( R \) is universal, and we have the contradiction we were looking for.

Let “true arithmetic” be the theory \( \{ \varphi : N \models \varphi \} \), that is, the set of sentences in the language of arithmetic that are true in the standard interpretation.

Corollary tcp.3. True arithmetic is not decidable.