

## tcp.1 Consistent Extensions of $\mathbf{Q}$ are Undecidable

Remember that a theory is *consistent* if it does not prove both  $\varphi$  and  $\neg\varphi$  for any formula  $\varphi$ . Since anything follows from a contradiction, an inconsistent theory is trivial: every sentence is provable. Clearly, if a theory is  $\omega$ -consistent, then it is consistent. But being consistent is a weaker requirement (i.e., there are theories that are consistent but not  $\omega$ -consistent.). We can weaken the assumption in ?? to simple consistency to obtain a stronger theorem.

**Lemma tcp.1.** *There is no “universal computable relation.” That is, there is no binary computable relation  $R(x, y)$ , with the following property: whenever  $S(y)$  is a unary computable relation, there is some  $k$  such that for every  $y$ ,  $S(y)$  is true if and only if  $R(k, y)$  is true.*

*Proof.* Suppose  $R(x, y)$  is a universal computable relation. Let  $S(y)$  be the relation  $\neg R(y, y)$ . Since  $S(y)$  is computable, for some  $k$ ,  $S(y)$  is equivalent to  $R(k, y)$ . But then we have that  $S(k)$  is equivalent to both  $R(k, k)$  and  $\neg R(k, k)$ , which is a contradiction.  $\square$

**Theorem tcp.2.** *Let  $\mathbf{T}$  be any consistent theory that includes  $\mathbf{Q}$ . Then  $\mathbf{T}$  is not decidable.*

*Proof.* Suppose  $\mathbf{T}$  is a consistent, decidable extension of  $\mathbf{Q}$ . We will obtain a contradiction by using  $\mathbf{T}$  to define a universal computable relation.

Let  $R(x, y)$  hold if and only if

$x$  codes a formula  $\theta(u)$ , and  $\mathbf{T}$  proves  $\theta(\bar{y})$ .

Since we are assuming that  $\mathbf{T}$  is decidable,  $R$  is computable. Let us show that  $R$  is universal. If  $S(y)$  is any computable relation, then it is representable in  $\mathbf{Q}$  (and hence  $\mathbf{T}$ ) by a formula  $\theta_S(u)$ . Then for every  $n$ , we have

$$\begin{aligned} S(\bar{n}) &\rightarrow T \vdash \theta_S(\bar{n}) \\ &\rightarrow R(\# \theta_S(u)^\#, n) \end{aligned}$$

and

$$\begin{aligned} \neg S(\bar{n}) &\rightarrow T \vdash \neg \theta_S(\bar{n}) \\ &\rightarrow T \not\vdash \theta_S(\bar{n}) \quad (\text{since } \mathbf{T} \text{ is consistent}) \\ &\rightarrow \neg R(\# \theta_S(u)^\#, n). \end{aligned}$$

That is, for every  $y$ ,  $S(y)$  is true if and only if  $R(\# \theta_S(u)^\#, y)$  is. So  $R$  is universal, and we have the contradiction we were looking for.  $\square$

Let “true arithmetic” be the theory  $\{\varphi : \mathbb{N} \models \varphi\}$ , that is, the set of sentences in the language of arithmetic that are true in the standard interpretation.

**Corollary tcp.3.** *True arithmetic is not decidable.*

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**Bibliography**