tcp.1 Consistent Extensions of Q are Undecidable

Remember that a theory is consistent if it does not prove both φ and ¬φ for any formula φ. Since anything follows from a contradiction, an inconsistent theory is trivial: every sentence is provable. Clearly, if a theory if ω-consistent, then it is consistent. But being consistent is a weaker requirement (i.e., there are theories that are consistent but not ω-consistent.) We can weaken the assumption in ?? to simple consistency to obtain a stronger theorem.

**Lemma tcp.1.** There is no “universal computable relation.” That is, there is no binary computable relation R(x, y), with the following property: whenever S(y) is a unary computable relation, there is some k such that for every y, S(y) is true if and only if R(k, y) is true.

*Proof.* Suppose R(x, y) is a universal computable relation. Let S(y) be the relation ¬R(y, y). Since S(y) is computable, for some k, S(y) is equivalent to R(k, y). But then we have that S(k) is equivalent to both R(k, k) and ¬R(k, k), which is a contradiction. □

**Theorem tcp.2.** Let T be any consistent theory that includes Q. Then T is not decidable.

*Proof.* Suppose T is a consistent, decidable extension of Q. We will obtain a contradiction by using T to define a universal computable relation.

Let R(x, y) hold if and only if
\[
x \text{ codes a formula } \theta(u), \text{ and } T \text{ proves } \theta(y).
\]

Since we are assuming that T is decidable, R is computable. Let us show that R is universal. If S(y) is any computable relation, then it is representable in Q (and hence T) by a formula θS(u). Then for every n, we have
\[
S(\pi) \rightarrow T \vdash \theta_S(\pi)
\]
\[
\rightarrow R(\#\theta_S(u)#, n)
\]

and
\[
\neg S(\pi) \rightarrow T \vdash \neg \theta_S(\pi)
\]
\[
\rightarrow T \not\vdash \theta_S(\pi) \quad \text{(since T is consistent)}
\]
\[
\rightarrow \neg R(\#\theta_S(u)#, n).
\]

That is, for every y, S(y) is true if and only if R(\#\theta_S(u)#, y) is. So R is universal, and we have the contradiction we were looking for. □

Let “true arithmetic” be the theory \{ φ : N \models φ \}, that is, the set of sentences in the language of arithmetic that are true in the standard interpretation.

**Corollary tcp.3.** True arithmetic is not decidable.
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Bibliography