tcp.1 Theories Consistent with Q are Undecidable

The following theorem says that not only is **Q** undecidable, but, in fact, any theory that does not disagree with **Q** is undecidable.

Theorem tcp.1. *Let* **T** *be any theory in the language of arithmetic that is consistent with* **Q** *(i.e.,* **T** $\cup$ **Q** *is consistent). Then* **T** *is undecidable.*

Proof. Remember that **Q** has a finite set of axioms, $Q_1, \ldots, Q_8$. We can even replace these by a single axiom, $\alpha = Q_1 \wedge \cdots \wedge Q_8$.

Suppose **T** is a decidable theory consistent with **Q**. Let

$$C = \{\varphi : \mathbf{T} \vdash \alpha \to \varphi\}.$$

We show that C would be a computable separation of **Q** and $\bar{\mathbf{Q}}$, a contradiction. First, if φ is in **Q**, then φ is provable from the axioms of **Q**; by the deduction theorem, there is a proof of $\alpha \to \varphi$ in first-order logic. So φ is in C.

On the other hand, if φ is in $\bar{\mathbf{Q}}$, then there is a proof of $\alpha \to \neg\varphi$ in first-order logic. If **T** also proves $\alpha \to \varphi$, then **T** proves $\neg\alpha$, in which case **T** $\cup$ **Q** is inconsistent. But we are assuming **T** $\cup$ **Q** is consistent, so **T** does not prove $\alpha \to \varphi$, and so φ is not in C.

We've shown that if φ is in **Q**, then it is in C, and if φ is in $\bar{\mathbf{Q}}$, then it is in $\overline{C}$. So C is a computable separation, which is the contradiction we were looking for. $\square$

This theorem is very powerful. For example, it implies:

Corollary tcp.2. *First-order logic for the language of arithmetic (that is, the set* $\{\varphi : \varphi$ *is provable in first-order logic*$\}$*) is undecidable.*

Proof. First-order logic is the set of consequences of $\emptyset$, which is consistent with **Q**. $\square$

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Bibliography