

## tcp.1 Theories Consistent with $\mathbf{Q}$ are Undecidable

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The following theorem says that not only is  $\mathbf{Q}$  undecidable, but, in fact, any theory that does not disagree with  $\mathbf{Q}$  is undecidable.

**Theorem tcp.1.** *Let  $\mathbf{T}$  be any theory in the language of arithmetic that is consistent with  $\mathbf{Q}$  (i.e.,  $\mathbf{T} \cup \mathbf{Q}$  is consistent). Then  $\mathbf{T}$  is undecidable.*

*Proof.* Remember that  $\mathbf{Q}$  has a finite set of axioms,  $Q_1, \dots, Q_8$ . We can even replace these by a single axiom,  $\alpha = Q_1 \wedge \dots \wedge Q_8$ .

Suppose  $\mathbf{T}$  is a decidable theory consistent with  $\mathbf{Q}$ . Let

$$C = \{\varphi : \mathbf{T} \vdash \alpha \rightarrow \varphi\}.$$

We show that  $C$  would be a computable separation of  $\mathbf{Q}$  and  $\bar{\mathbf{Q}}$ , a contradiction. First, if  $\varphi$  is in  $\mathbf{Q}$ , then  $\varphi$  is provable from the axioms of  $\mathbf{Q}$ ; by the deduction theorem, there is a proof of  $\alpha \rightarrow \varphi$  in first-order logic. So  $\varphi$  is in  $C$ .

On the other hand, if  $\varphi$  is in  $\bar{\mathbf{Q}}$ , then there is a proof of  $\alpha \rightarrow \neg\varphi$  in first-order logic. If  $\mathbf{T}$  also proves  $\alpha \rightarrow \varphi$ , then  $\mathbf{T}$  proves  $\neg\alpha$ , in which case  $\mathbf{T} \cup \mathbf{Q}$  is inconsistent. But we are assuming  $\mathbf{T} \cup \mathbf{Q}$  is consistent, so  $\mathbf{T}$  does not prove  $\alpha \rightarrow \varphi$ , and so  $\varphi$  is not in  $C$ .

We've shown that if  $\varphi$  is in  $\mathbf{Q}$ , then it is in  $C$ , and if  $\varphi$  is in  $\bar{\mathbf{Q}}$ , then it is in  $\bar{C}$ . So  $C$  is a computable separation, which is the contradiction we were looking for.  $\square$

This theorem is very powerful. For example, it implies:

**Corollary tcp.2.** *First-order logic for the language of arithmetic (that is, the set  $\{\varphi : \varphi \text{ is provable in first-order logic}\}$ ) is undecidable.*

*Proof.* First-order logic is the set of consequences of  $\emptyset$ , which is consistent with  $\mathbf{Q}$ .  $\square$

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## Bibliography