tcp.1 Theories Consistent with Q are Undecidable

inc:tcp:con:sec The following theorem says that not only is $\mathbf{Q}$ undecidable, but, in fact, any theory that does not disagree with $\mathbf{Q}$ is undecidable.

Theorem tcp.1. *Let* $\mathbf{T}$ *be any theory in the language of arithmetic that is consistent with* $\mathbf{Q}$ *(i.e.,* $\mathbf{T} \cup \mathbf{Q}$ *is consistent). Then* $\mathbf{T}$ *is undecidable.*

Proof. Remember that $\mathbf{Q}$ has a finite set of axioms, $Q_1, \dots, Q_8$. We can even replace these by a single axiom, $\alpha = Q_1 \land \dots \land Q_8$.

Suppose $\mathbf{T}$ is a decidable theory consistent with $\mathbf{Q}$. Let

$$C = \{\varphi : \mathbf{T} \vdash \alpha \rightarrow \varphi\}.$$

We show that C would be a computable separation of $\mathbf{Q}$ and $\bar{\mathbf{Q}}$, a contradiction. First, if φ is in $\mathbf{Q}$, then φ is provable from the axioms of $\mathbf{Q}$; by the deduction theorem, there is a derivation of $\alpha \rightarrow \varphi$ in first-order logic. So φ is in C.

On the other hand, if φ is in $\bar{\mathbf{Q}}$, then there is a proof of $\alpha \rightarrow \lnot\varphi$ in first-order logic. If $\mathbf{T}$ also proves $\alpha \rightarrow \varphi$, then $\mathbf{T}$ proves $\lnot\alpha$, in which case $\mathbf{T} \cup \mathbf{Q}$ is inconsistent. But we are assuming $\mathbf{T} \cup \mathbf{Q}$ is consistent, so $\mathbf{T}$ does not prove $\alpha \rightarrow \varphi$, and so φ is not in C.

We've shown that if φ is in $\mathbf{Q}$, then it is in C, and if φ is in $\bar{\mathbf{Q}}$, then it is in $\overline{C}$. So C is a computable separation, which is the contradiction we were looking for. □

This theorem is very powerful. For example, it implies:

Corollary tcp.2. *First-order logic for the language of arithmetic (that is, the set* $\{\varphi : \varphi$ *is provable in first-order logic*$\}$*) is undecidable.*

Proof. First-order logic is the set of consequences of $\emptyset$, which is consistent with $\mathbf{Q}$. □

Photo Credits

Bibliography