tcp.1 Axiomatizable Theories

A theory $T$ is said to be **axiomatizable** if it has a computable set of axioms $A$. (Saying that $A$ is a set of axioms for $T$ means $T = \{ \varphi : A \vdash \varphi \}$.) Any “reasonable” axiomatization of the natural numbers will have this property. In particular, any theory with a finite set of axioms is axiomatizable.

**Lemma tcp.1.** Suppose $T$ is axiomatizable. Then $T$ is computably enumerable.

*Proof.* Suppose $A$ is a computable set of axioms for $T$. To determine if $\varphi \in T$, just search for a proof of $\varphi$ from the axioms.

Put slightly differently, $\varphi$ is in $T$ if and only if there is a finite list of axioms $\psi_1, \ldots, \psi_k$ in $A$ and a proof of $(\psi_1 \land \cdots \land \psi_k) \rightarrow \varphi$ in first-order logic. But we already know that any set with a definition of the form “there exists . . . such that . . .” is c.e., provided the second “. . .” is computable. \qed

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Bibliography