

## tcp.1 **Axiomatizable** Complete Theories are Decidable

inc:tcp:edc:  
sec A theory is said to be *complete* if for every sentence  $\varphi$ , either  $\varphi$  or  $\neg\varphi$  is provable.

**Lemma tcp.1.** *Suppose a theory  $\mathbf{T}$  is complete and **axiomatizable**. Then  $\mathbf{T}$  is decidable.*

*Proof.* Suppose  $\mathbf{T}$  is complete and  $A$  is a computable set of axioms. If  $\mathbf{T}$  is inconsistent, it is clearly computable. (Algorithm: “just say yes.”) So we can assume that  $\mathbf{T}$  is also consistent.

To decide whether or not a sentence  $\varphi$  is in  $\mathbf{T}$ , simultaneously search for a proof of  $\varphi$  from  $A$  and a proof of  $\neg\varphi$ . Since  $\mathbf{T}$  is complete, you are bound to find one or another; and since  $\mathbf{T}$  is consistent, if you find a proof of  $\neg\varphi$ , there is no proof of  $\varphi$ .

Put in different terms, we already know that  $\mathbf{T}$  is **c.e.**; so by a theorem we proved before, it suffices to show that the complement of  $\mathbf{T}$  is **c.e.** also. But a formula  $\varphi$  is in  $\bar{\mathbf{T}}$  if and only if  $\neg\varphi$  is in  $\mathbf{T}$ ; so  $\bar{\mathbf{T}} \leq_m \mathbf{T}$ .  $\square$

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## Bibliography