tcp.1  **Axiomatizable Complete Theories are Decidable**

A theory is said to be *complete* if for every sentence \( \varphi \), either \( \varphi \) or \( \neg \varphi \) is provable.

**Lemma tcp.1.**  *Suppose a theory \( T \) is complete and axiomatizable. Then \( T \) is decidable.*

*Proof.* Suppose \( T \) is complete and \( A \) is a computable set of axioms. If \( T \) is inconsistent, it is clearly computable. (Algorithm: “just say yes.”) So we can assume that \( T \) is also consistent.

To decide whether or not a sentence \( \varphi \) is in \( T \), simultaneously search for a derivation of \( \varphi \) from \( T \) and a derivation of \( \neg \varphi \). Since \( T \) is complete, you are bound to find one or the other; and since \( T \) is consistent, if you find a derivation of \( \neg \varphi \), there is no derivation of \( \varphi \).

Put in different terms, we already know that \( T \) is c.e.; so by a theorem we proved before, it suffices to show that the complement of \( T \) is c.e. also. But a formula \( \varphi \) is in \( \bar{T} \) if and only if \( \neg \varphi \) is in \( T \); so \( \bar{T} \leq_m T \).

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**Bibliography**