Lemma tcp.1. Suppose a theory $T$ is complete and axiomatizable. Then $T$ is decidable.

Proof. Suppose $T$ is complete and $A$ is a computable set of axioms. If $T$ is inconsistent, it is clearly computable. (Algorithm: “just say yes.”) So we can assume that $T$ is also consistent.

To decide whether or not a sentence $\varphi$ is in $T$, simultaneously search for a derivation of $\varphi$ from $T$ and a derivation of $\neg \varphi$. Since $T$ is complete, you are bound to find one or the other; and since $T$ is consistent, if you find a derivation of $\neg \varphi$, there is no derivation of $\varphi$.

Put in different terms, we already know that $T$ is c.e.; so by a theorem we proved before, it suffices to show that the complement of $T$ is c.e. also. But a formula $\varphi$ is in $\bar{T}$ if and only if $\neg \varphi$ is in $T$; so $\bar{T} \leq_m T$. 

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Bibliography