tcp.1 Axiomatizable Complete Theories are Decidable

inc:tcp:cdc:sec A theory is said to be *complete* if for every sentence φ, either φ or $\neg\varphi$ is provable.

Lemma tcp.1. *Suppose a theory* $\mathbf{T}$ *is complete and axiomatizable. Then* $\mathbf{T}$ *is decidable.*

Proof. Suppose $\mathbf{T}$ is complete and A is a computable set of axioms. If $\mathbf{T}$ is inconsistent, it is clearly computable. (Algorithm: "just say yes.") So we can assume that $\mathbf{T}$ is also consistent.

To decide whether or not a sentence φ is in $\mathbf{T}$, simultaneously search for a proof of φ from A and a proof of $\neg\varphi$. Since $\mathbf{T}$ is complete, you are bound to find one or another; and since $\mathbf{T}$ is consistent, if you find a proof of $\neg\varphi$, there is no proof of φ.

Put in different terms, we already know that $\mathbf{T}$ is c.e.; so by a theorem we proved before, it suffices to show that the complement of $\mathbf{T}$ is c.e. also. But a formula φ is in $\bar{\mathbf{T}}$ if and only if $\neg\varphi$ is in $\mathbf{T}$; so $\bar{\mathbf{T}} \leq_m \mathbf{T}$. □

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Bibliography