

req.1 Undecidability

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We call a theory \mathbf{T} *undecidable* if there is no computational procedure which, after finitely many steps and unfailingly, provides a correct answer to the question “does \mathbf{T} prove φ ?” for any sentence φ in the language of \mathbf{T} . So \mathbf{Q} would be decidable iff there were a computational procedure which decides, given a sentence φ in the language of arithmetic, whether $\mathbf{Q} \vdash \varphi$ or not. We can make this more precise by asking: Is the relation $\text{Prov}_{\mathbf{Q}}(y)$, which holds of y iff y is the Gödel number of a sentence provable in \mathbf{Q} , recursive? The answer is: no.

Theorem req.1. \mathbf{Q} is undecidable, i.e., the relation

$$\text{Prov}_{\mathbf{Q}}(y) \Leftrightarrow \text{Sent}(y) \wedge \exists x \text{Prf}_{\mathbf{Q}}(x, y)$$

is not recursive.

Proof. Suppose it were. Then we could solve the halting problem as follows: Given e and n , we know that $\varphi_e(n) \downarrow$ iff there is an s such that $T(e, n, s)$, where T is Kleene’s predicate from ???. Since T is primitive recursive it is representable in \mathbf{Q} by a formula ψ_T , that is, $\mathbf{Q} \vdash \psi_T(\bar{e}, \bar{n}, \bar{s})$ iff $T(e, n, s)$. If $\mathbf{Q} \vdash \psi_T(\bar{e}, \bar{n}, \bar{s})$ then also $\mathbf{Q} \vdash \exists y \psi_T(\bar{e}, \bar{n}, y)$. If no such s exists, then $\mathbf{Q} \vdash \neg \psi_T(\bar{e}, \bar{n}, \bar{s})$ for every s . But \mathbf{Q} is ω -consistent, i.e., if $\mathbf{Q} \vdash \neg \varphi(\bar{n})$ for every $n \in \mathbb{N}$, then $\mathbf{Q} \not\vdash \exists y \varphi(y)$. We know this because the axioms of \mathbf{Q} are true in the standard model \mathfrak{N} . So, $\mathbf{Q} \not\vdash \exists y \psi_T(\bar{e}, \bar{n}, y)$. In other words, $\mathbf{Q} \vdash \exists y \psi_T(\bar{e}, \bar{n}, y)$ iff there is an s such that $T(e, n, s)$, i.e., iff $\varphi_e(n) \downarrow$. From e and n we can compute $\# \exists y \psi_T(\bar{e}, \bar{n}, y)^\#$, let $g(e, n)$ be the primitive recursive function which does that. So

$$h(e, n) = \begin{cases} 1 & \text{if } \text{Prov}_{\mathbf{Q}}(g(e, n)) \\ 0 & \text{otherwise.} \end{cases}$$

This would show that h is recursive if $\text{Prov}_{\mathbf{Q}}$ is. But h is not recursive, by ??, so $\text{Prov}_{\mathbf{Q}}$ cannot be either. \square

Corollary req.2. *First-order logic is undecidable.*

Proof. If first-order logic were decidable, provability in \mathbf{Q} would be as well, since $\mathbf{Q} \vdash \varphi$ iff $\vdash \omega \rightarrow \varphi$, where ω is the conjunction of the axioms of \mathbf{Q} . \square

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Bibliography