inp.1 Σ_1 completeness

inc:inp:s1c: sec

Despite the incompleteness of $\mathbf{Q}$ and its consistent, axiomatizable extensions, we have seen that $\mathbf{Q}$ does prove many basic facts about numerals. In fact, this can be extended quite considerably. To understand the scope of what can be proved in $\mathbf{Q}$, we introduce the notions of Δ_0, Σ_1, and Π_1 formulas. Roughly speaking, a Σ_1 formula is one of the form $\exists x\, \psi(x)$, where ψ is constructed using only propositional connectives and bounded quantifiers. We shall show that if φ is a Σ_1 sentence which is true in $\mathfrak{N}$, then $\mathbf{Q} \vdash \varphi$ (Theorem inp.7).

inc:inp:s1c: defn:bd-quant

Definition inp.1. A *bounded existential formula* is one of the form $\exists x\,(x < t \land \varphi(x))$ where t is any term, which we conventionally write as $(\exists x < t)\ \varphi(x)$. A *bounded universal formula* is one of the form $\forall x\,(x < t \to \varphi(x))$ where t is any term, which we conventionally write as $(\forall x < t)\ \varphi(x)$.

inc:inp:s1c: defn:delta0-sigma1-pi1-frm

Definition inp.2. A formula ψ is Δ_0 if it is built up from atomic formulas using only propositional connectives and bounded quantification. A formula φ is Σ_1 if $\varphi \equiv \exists x\, \psi(x)$ where ψ is Δ_0. A formula φ is Π_1 if $\varphi \equiv \forall x\, \psi(x)$ where ψ is Δ_0.

inc:inp:s1c: lem:q-proves-clterm-id

Lemma inp.3. *Suppose t is a closed term such that $\mathrm{Val}^{\mathfrak{N}}(t) = n$. Then $\mathbf{Q} \vdash t = \overline{n}$.*

Proof. We prove this by induction on the complexity of t. For the base case, $\mathrm{Val}^{\mathfrak{N}}(\mathsf{o}) = 0$, and $\mathbf{Q} \vdash \mathsf{o} = \overline{0}$ since $\overline{0} \equiv \mathsf{o}$. For the inductive case, let t_1 and t_2 be terms such that $\mathrm{Val}^{\mathfrak{N}}(t_1) = n_1$, $\mathrm{Val}^{\mathfrak{N}}(t_2) = n_2$, $\mathbf{Q} \vdash t_1 = \overline{n}_1$, and $\mathbf{Q} \vdash t_2 = \overline{n}_2$.

Then $\mathrm{Val}^{\mathfrak{N}}((t_1')) = n_1 + 1$, and we have that $\mathbf{Q} \vdash t_1' = \overline{n}_1{}'$ by the first-order rules for identity applied to the induction hypothesis and the formula $\overline{n_1}' = \overline{n_1}'$, so we have $\mathbf{Q} \vdash t_1' = \overline{n_1 + 1}$ by the definition of numerals.

For sums we have

$$\mathrm{Val}^{\mathfrak{N}}((t_1 + t_2)) = \mathrm{Val}^{\mathfrak{N}}(t_1) + \mathrm{Val}^{\mathfrak{N}}(t_2) = n_1 + n_2.$$

By the induction hypothesis and the rules for identity, $\mathbf{Q} \vdash t_1 + t_2 = \overline{n_1} + t_2$, and then $\mathbf{Q} \vdash t_1 + t_2 = \overline{n_1} + \overline{n_2}$ by a second application of the rules for identity. By **??**, $\mathbf{Q} \vdash \overline{n_1} + \overline{n_2} = \overline{n_1 + n_2}$, so $\mathbf{Q} \vdash t_1 + t_2 = \overline{n_1 + n_2}$.

Similar reasoning also works for $\times$, using **??**. Since this exhausts the closed terms of arithmetic, we have that $\mathbf{Q} \vdash t = \overline{n}$ for all closed terms t such that $\mathrm{Val}^{\mathfrak{N}}(t) = n$. □

Problem inp.1. Prove in detail the part of Lemma inp.3 involving $\times$.

inc:inp:s1c: lem:atomic-completeness

Lemma inp.4. *Suppose t_1 and t_2 are closed terms. Then*

1. *If $\mathrm{Val}^{\mathfrak{N}}(t_1) = \mathrm{Val}^{\mathfrak{N}}(t_2)$, then $\mathbf{Q} \vdash t_1 = t_2$.*
2. *If $\mathrm{Val}^{\mathfrak{N}}(t_1) \neq \mathrm{Val}^{\mathfrak{N}}(t_2)$, then $\mathbf{Q} \vdash t_1 \neq t_2$.*
3. *If $\mathrm{Val}^{\mathfrak{N}}(t_1) < \mathrm{Val}^{\mathfrak{N}}(t_2)$, then $\mathbf{Q} \vdash t_1 < t_2$.*

4. *If* $\mathrm{Val}^{\mathfrak{N}}(t_2) \le \mathrm{Val}^{\mathfrak{N}}(t_1)$*, then* $\mathbf{Q} \vdash \lnot(t_1 < t_2)$.

Proof. Given terms t_1 and t_2, we fix $n = \mathrm{Val}^{\mathfrak{N}}(t_1)$ and $m = \mathrm{Val}^{\mathfrak{N}}(t_2)$.

Suppose $\varphi \equiv t_1 = t_2$. By Lemma inp.3, $\mathbf{Q} \vdash t_1 = \overline{n}$ and $\mathbf{Q} \vdash t_2 = \overline{n}$. If $n = m$, then $\mathbf{Q} \vdash \overline{n} = \overline{m}$ and hence $\mathbf{Q} \vdash t_1 = t_2$ by the transitivity of identity. If $n \neq m$ then $\mathbf{Q} \vdash \overline{n} \neq \overline{m}$, and by the transitivity of identity again, $\mathbf{Q} \vdash t_1 \neq t_2$.

Now let $\varphi \equiv t_1 < t_2$. For both cases, we rely on axiom Q_8, which states that $x < y \leftrightarrow \exists z\, z' + x = y$ for all x, y.

Suppose $\mathfrak{N} \vDash t_1 < t_2$. Then there exists some $k \in \mathbb{N}$ such that $n+k+1 = m$. By Lemma inp.3, $\mathbf{Q} \vdash t_1 = \overline{n}$ and $\mathbf{Q} \vdash t_2 = \overline{m}$, and by the first part of this lemma, $\mathbf{Q} \vdash \overline{n} + \overline{k}' = \overline{m}$. By the transitivity of identity it follows that $\mathbf{Q} \vdash \overline{k}' + t_1 = t_2$, so $\mathbf{Q} \vdash \exists z\, z' + t_1 = t_2$. By the right-to-left direction of Q_8, $\mathbf{Q} \vdash t_1 < t_2$.

Suppose instead that $\mathfrak{N} \nvDash t_1 < t_2$, i.e., $m \le n$. We work in $\mathbf{Q}$ and assume that $t_1 < t_2$. By the left-to-right direction of Q_8, there is some z such that $z' + t_1 = t_2$. Since $\mathbf{Q} \vdash t_1 = \overline{n}$ and $\mathbf{Q} \vdash t_2 = \overline{m}$, $z' + \overline{n} = \overline{m}$. By an external induction on m using Q_5, $z' + \overline{n - m} = \mathsf{o}$. If $m = n$ then $z' \neq \mathsf{o}$, giving a contradiction via Q_3. If $m < n$ then $(z' + \overline{n - m - 1})' = \mathsf{o}$ by Q_5 again, giving a contradiction via Q_3. So $\mathbf{Q} \vdash \lnot(t_1 < t_2)$. □

Lemma inp.5. *Suppose* φ *is a formula,* t *a closed term, and* $k = \mathrm{Val}^{\mathfrak{N}}(t)$. *Then* inc:inp:s1c: lem:bounded-quant-equiv

1. $\mathbf{Q} \vdash (\forall x < t)\, \varphi(x)$ *iff* $\mathbf{Q} \vdash \varphi(\overline{0}) \land \dots \land \varphi(\overline{k-1})$.

2. $\mathbf{Q} \vdash (\exists x < t)\, \varphi(x)$ *iff* $\mathbf{Q} \vdash \varphi(\overline{0}) \lor \dots \lor \varphi(\overline{k-1})$.

Proof. We prove the case for the bounded universal quantifier. If $\mathrm{Val}^{\mathfrak{N}}(t) = 0$ then the left-hand side of the equivalence is provable in $\mathbf{Q}$, because there is no $x < \overline{0}$ by **??**. Similarly, we can take an empty disjunction to be simply $\top$, which is also provable in $\mathbf{Q}$. We therefore suppose that $\mathrm{Val}^{\mathfrak{N}}(t) = k + 1$ for some natural number k. By Lemma inp.3 we can assume that we are working with a formula of the form $(\forall x < \overline{k+1})\, \varphi(x)$.

Suppose that $\mathbf{Q} \vdash (\forall x < \overline{k+1})\, \varphi(x)$, and let $n \le k$. Since $\mathbf{Q} \vdash \overline{n} < \overline{k+1}$ by Lemma inp.4, it follows by logic that $\mathbf{Q} \vdash \varphi(\overline{n})$. Applying this fact $k+1$ times for each $n \le k$, we get that $\mathbf{Q} \vdash \varphi(\overline{0}) \land \dots \land \varphi(\overline{k})$ as desired.

For the other direction, suppose that $\mathbf{Q} \vdash \varphi(\overline{0}) \land \dots \land \varphi(\overline{k})$. Working in $\mathbf{Q}$, suppose that $x < \overline{k+1}$. By **??** we have that $x = \overline{0} \lor \dots \lor x = \overline{k}$, so by logic it follows that $\varphi(x)$, and hence the universal claim $(\forall x < \overline{k+1})\, \varphi(x)$ follows.

The proof of the equivalence for bounded existentially quantified formulas is similar. □

Problem inp.2. Give a detailed proof of the existential case in Lemma inp.5.

Lemma inp.6. *If* φ *is a* Δ_0 *sentence which is true in* $\mathfrak{N}$*, then* $\mathbf{Q} \vdash \varphi$. inc:inp:s1c: lem:delta0-completeness

Proof. We prove this by induction on formula complexity. The base case is given by Lemma inp.4, so we move to the induction step. For simplicity we split the case of negation into subcases depending on the structure of the formula to which the negation is applied.

1. Suppose $(\varphi \land \psi)$ is true in $\mathfrak{N}$, so φ and ψ are true in $\mathfrak{N}$. By the induction hypothesis, $\mathbf{Q} \vdash \varphi$ and $\mathbf{Q} \vdash \psi$, so $\mathbf{Q} \vdash (\varphi \land \psi)$ by logic.

2. Suppose $\lnot(\varphi \land \psi)$ is true in $\mathfrak{N}$, so either $\lnot\varphi$ or $\lnot\psi$ is true in $\mathfrak{N}$. Without loss of generality, suppose the former. By the induction hypothesis $\mathbf{Q} \vdash \lnot\varphi$, and hence $\mathbf{Q} \vdash \lnot(\varphi \land \psi)$ by logic.

3. Suppose $(\varphi \lor \psi)$ is true in $\mathfrak{N}$, so either φ is true in $\mathfrak{N}$ or ψ is true in $\mathfrak{N}$. Without loss of generality, suppose the former holds. By the induction hypothesis $\mathbf{Q} \vdash \varphi$, and hence $\mathbf{Q} \vdash (\varphi \lor \psi)$ by logic.

4. Suppose $\lnot(\varphi \lor \psi)$ is true in $\mathfrak{N}$, so $\lnot\varphi$ and $\lnot\psi$ are true in $\mathfrak{N}$. Then $\mathbf{Q} \vdash \lnot\varphi$ and $\mathbf{Q} \vdash \lnot\psi$ by the induction hypothesis. Consequently, $\mathbf{Q} \vdash \lnot(\varphi \lor \psi)$ by logic.

5. Suppose that $(\forall x < t)\ \varphi(x)$ is true in $\mathfrak{N}$, where t is a closed term and $k = \mathrm{Val}^{\mathfrak{N}}(t)$. By the induction hypothesis and logic, if $\varphi(\overline{n})$ is true in $\mathfrak{N}$ for all $n < \mathrm{Val}^{\mathfrak{N}}(t)$ then $\mathbf{Q} \vdash \varphi(\overline{0}) \land \dots \land \varphi(\overline{k-1})$. By Lemma inp.5 it follows that $\mathbf{Q} \vdash (\forall x < t)\ \varphi(x)$.

6. The case for the bounded existential quantifier, where we have a sentence of the form $(\exists x < t)\ \varphi(x)$, is similar to that for the bounded universal quantifier.

7. Suppose that $\lnot(\forall x < t)\ \varphi(x)$ is true in $\mathfrak{N}$, where t is a closed term. This sentence is equivalent to the sentence $(\exists x < t)\ \lnot\varphi(x)$, with the equivalence derivable in $\mathbf{Q}$, so we may apply the reasoning for bounded existential quantifiers.

8. Similarly, suppose that $\lnot(\exists x < t)\ \varphi(x)$ is true in $\mathfrak{N}$, where t is a closed term. This sentence is equivalent in $\mathbf{Q}$ to $(\forall x < t)\ \lnot\varphi(x)$, and so we may apply the reasoning for bounded universal quantifiers.

9. Finally, suppose $\lnot\varphi$ is true in $\mathfrak{N}$. The only cases remaining are when φ is atomic and when $\lnot\varphi \equiv \lnot\lnot\psi$ for some Δ_0 sentence ψ. If φ is atomic then by Lemma inp.4, $\mathbf{Q} \vdash \lnot\varphi$. If $\lnot\varphi \equiv \lnot\lnot\psi$, then by logic it is provably equivalent in $\mathbf{Q}$ to ψ, which is true in $\mathfrak{N}$ since $\lnot\varphi$ is true in $\mathfrak{N}$. By the induction hypothesis we therefore have that $\mathbf{Q} \vdash \lnot\varphi$. □

Problem inp.3. Give a detailed proof of the existential case in Lemma inp.6.

inc:inp:s1c: thm:sigma1-completeness

Theorem inp.7. *If φ is a Σ_1 sentence which is true in $\mathfrak{N}$, then $\mathbf{Q} \vdash \varphi$.*

Proof. If $\exists x \varphi(x)$ is a Σ_1 sentence which is true in $\mathfrak{N}$, then there exists a natural number n and a variable assignment s such that $s(x) = n$ and $\mathfrak{N}, s \vDash \varphi(x)$. By standard facts about the satisfaction relation it follows that $\mathfrak{N} \vDash \varphi(\overline{n})$. But $\varphi(\overline{n})$ is a Δ_0 formula, so by Lemma inp.6 we have that $\mathbf{Q} \vdash \varphi(\overline{n})$, and hence by logic we also have that $\mathbf{Q} \vdash \exists x\, \varphi(x)$. □

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