Representing Relations

Let us say what it means for a relation to be representable.

**Definition req.1.** A relation $R(x_0, \ldots, x_k)$ on the natural numbers is representable in $\mathbb{Q}$ if there is a formula $\varphi_R(x_0, \ldots, x_k)$ such that whenever $R(n_0, \ldots, n_k)$ is true, $\mathbb{Q}$ proves $\varphi_R(n_0, \ldots, n_k)$, and whenever $R(n_0, \ldots, n_k)$ is false, $\mathbb{Q}$ proves $\neg\varphi_R(n_0, \ldots, n_k)$.

**Theorem req.2.** A relation is representable in $\mathbb{Q}$ if and only if it is computable.

*Proof.* For the forwards direction, suppose $R(x_0, \ldots, x_k)$ is represented by the formula $\varphi_R(x_0, \ldots, x_k)$. Here is an algorithm for computing $R$: on input $n_0, \ldots, n_k$, simultaneously search for a proof of $\varphi_R(n_0, \ldots, n_k)$ and a proof of $\neg\varphi_R(n_0, \ldots, n_k)$. By our hypothesis, the search is bound to find one or the other; if it is the first, report “yes,” and otherwise, report “no.”

In the other direction, suppose $R(x_0, \ldots, x_k)$ is computable. By definition, this means that the function $\chi_R(x_0, \ldots, x_k)$ is computable. By ??, $\chi_R$ is represented by a formula, say $\varphi_{\chi_R}(x_0, \ldots, x_k, y)$. Let $\varphi_R(x_0, \ldots, x_k)$ be the formula $\varphi_{\chi_R}(x_0, \ldots, x_k, 1)$. Then for any $n_0, \ldots, n_k$, if $R(n_0, \ldots, n_k)$ is true, then $\chi_R(n_0, \ldots, n_k) = 1$, in which case $\mathbb{Q}$ proves $\varphi_{\chi_R}(n_0, \ldots, n_k, 1)$, and so $\mathbb{Q}$ proves $\varphi_R(n_0, \ldots, n_k)$. On the other hand, if $R(n_0, \ldots, n_k)$ is false, then $\chi_R(n_0, \ldots, n_k) = 0$. This means that $\mathbb{Q}$ proves

$$\forall y (\varphi_{\chi_R}(n_0, \ldots, n_k, y) \to y = 1).$$

Since $\mathbb{Q}$ proves $\bar{1} \neq 1$, $\mathbb{Q}$ proves $\neg\varphi_{\chi_R}(n_0, \ldots, n_k, 1)$, and so it proves $\neg\varphi_R(n_0, \ldots, n_k)$.

**Problem req.1.** Show that if $R$ is representable in $\mathbb{Q}$, so is $\chi_R$.

Photo Credits

Bibliography