

## req.1 Representing Relations

inc:req:rel:  
sec Let us say what it means for a *relation* to be representable.

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defn:representing-relations **Definition req.1.** A relation  $R(x_0, \dots, x_k)$  on the natural numbers is *representable in  $\mathbf{Q}$*  if there is a formula  $\varphi_R(x_0, \dots, x_k)$  such that whenever  $R(n_0, \dots, n_k)$  is true,  $\mathbf{Q}$  proves  $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$ , and whenever  $R(n_0, \dots, n_k)$  is false,  $\mathbf{Q}$  proves  $\neg\varphi_R(\overline{n_0}, \dots, \overline{n_k})$ .

inc:req:rel:  
thm:representing-rels **Theorem req.2.** *A relation is representable in  $\mathbf{Q}$  if and only if it is computable.*

*Proof.* For the forwards direction, suppose  $R(x_0, \dots, x_k)$  is represented by the formula  $\varphi_R(x_0, \dots, x_k)$ . Here is an algorithm for computing  $R$ : on input  $n_0, \dots, n_k$ , simultaneously search for a proof of  $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$  and a proof of  $\neg\varphi_R(\overline{n_0}, \dots, \overline{n_k})$ . By our hypothesis, the search is bound to find one or the other; if it is the first, report “yes,” and otherwise, report “no.”

In the other direction, suppose  $R(x_0, \dots, x_k)$  is computable. By definition, this means that the function  $\chi_R(x_0, \dots, x_k)$  is computable. By ??,  $\chi_R$  is represented by a formula, say  $\varphi_{\chi_R}(x_0, \dots, x_k, y)$ . Let  $\varphi_R(x_0, \dots, x_k)$  be the formula  $\varphi_{\chi_R}(x_0, \dots, x_k, \overline{1})$ . Then for any  $n_0, \dots, n_k$ , if  $R(n_0, \dots, n_k)$  is true, then  $\chi_R(n_0, \dots, n_k) = 1$ , in which case  $\mathbf{Q}$  proves  $\varphi_{\chi_R}(\overline{n_0}, \dots, \overline{n_k}, \overline{1})$ , and so  $\mathbf{Q}$  proves  $\varphi_R(\overline{n_0}, \dots, \overline{n_k})$ . On the other hand, if  $R(n_0, \dots, n_k)$  is false, then  $\chi_R(n_0, \dots, n_k) = 0$ . This means that  $\mathbf{Q}$  proves

$$\forall y (\varphi_{\chi_R}(\overline{n_0}, \dots, \overline{n_k}, y) \rightarrow y = \overline{0}).$$

Since  $\mathbf{Q}$  proves  $\overline{0} \neq \overline{1}$ ,  $\mathbf{Q}$  proves  $\neg\varphi_{\chi_R}(\overline{n_0}, \dots, \overline{n_k}, \overline{1})$ , and so it proves  $\neg\varphi_R(\overline{n_0}, \dots, \overline{n_k})$ . □

**Problem req.1.** Show that if  $R$  is representable in  $\mathbf{Q}$ , so is  $\chi_R$ .

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## Bibliography