req.1 Functions Representable in Q are Computable

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Lemma req.1. *Every function that is representable in* **Q** *is computable.*

Proof. Let's first give the intuitive idea for why this is true. If $f(x_0, \dots, x_k)$ is representable in $\mathbf{Q}$, there is a formula $\varphi(x_0, \dots, x_k, y)$ such that

$$\mathbf{Q} \vdash \varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{m}) \quad \text{iff} \quad m = f(n_0, \dots, n_k).$$

To compute f, we do the following. List all the possible derivations δ in the language of arithmetic. This is possible to do mechanically. For each one, check if it is a derivation of a formula of the form $\varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{m})$. If it is, m must be $= f(n_0, \dots, n_k)$ and we've found the value of f. The search terminates because $\mathbf{Q} \vdash \varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{f(n_0, \dots, n_k)})$, so eventually we find a δ of the right sort.

This is not quite precise because our procedure operates on derivations and formulas instead of just on numbers, and we haven't explained exactly why "listing all possible derivations" is mechanically possible. But as we've seen, it is possible to code terms, formulas, and derivations by Gödel numbers. We've also introduced a precise model of computation, the general recursive functions. And we've seen that the relation $\mathrm{Prf}_{\mathbf{Q}}(d, y)$, which holds iff d is the Gödel number of a derivation of the formula with Gödel number x from the axioms of $\mathbf{Q}$, is (primitive) recursive. Other primitive recursive functions we'll need are num (**??**) and Subst (**??**). From these, it is possible to define f by minimization; thus, f is recursive.

First, define

$$\begin{aligned} A(n_0, \dots, n_k, m) = \\ & \mathrm{Subst}(\mathrm{Subst}(\dots \mathrm{Subst}({}^{\#}\varphi_f{}^{\#}, \mathrm{num}(n_0), {}^{\#}x_0{}^{\#}), \\ & \qquad \dots), \mathrm{num}(n_k), {}^{\#}x_k{}^{\#}), \mathrm{num}(m), {}^{\#}y^{\#}) \end{aligned}$$

This looks complicated, but it's just the function $A(n_0, \dots, n_k, m) = {}^{\#}\varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{m})^{\#}$.

Now, consider the relation $R(n_0, \dots, n_k, s)$ which holds if $(s)_0$ is the Gödel number of a derivation from $\mathbf{Q}$ of $\varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{(s)_1})$:

$$R(n_0, \dots, n_k, s) \quad \text{iff} \quad \mathrm{Prf}_{\mathbf{Q}}((s)_0, A(n_0, \dots, n_k, (s)_1)$$

If we can find an s such that $R(n_0, \dots, n_k, s)$ hold, we have found a pair of numbers—$(s)_0$ and (s_1)—such that $(s)_0$ is the Gödel number of a derivation of $A_f(\overline{n_0}, \dots, \overline{n_k}, (s)_1)$. So looking for s is like looking for the pair d and m in the informal proof. And a computable function that "looks for" such an s can be defined by regular minimization. Note that R is regular: for every $n_0, \dots, n_k$, there is a derivation δ of $\mathbf{Q} \vdash \varphi_f(\overline{n_0}, \dots, \overline{n_k}, \overline{f(n_0, \dots, n_k)})$, so $R(n_0, \dots, n_k, s)$ holds for $s = \langle {}^{\#}\delta^{\#}, f(n_0, \dots, n_k) \rangle$. So, we can write f as

$$f(n_0, \dots, n_k) = (\mu s\; R(n_0, \dots, n_k, s))_1.$$

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Bibliography