

req.1 Simulating Primitive Recursion

inc:req:pri: Now we can show that definition by primitive recursion can be “simulated”
sec by regular minimization using the beta function. Suppose we have $f(\vec{x})$ and $g(\vec{x}, y, z)$. Then the function $h(x, \vec{z})$ defined from f and g by primitive recursion is

$$\begin{aligned}h(\vec{x}, y) &= f(\vec{z}) \\h(\vec{x}, y + 1) &= g(\vec{x}, y, h(\vec{x}, y)).\end{aligned}$$

We need to show that h can be defined from f and g using just composition and regular minimization, using the basic functions and functions defined from them using composition and regular minimization (such as β).

inc:req:pri: **Lemma req.1.** *If h can be defined from f and g using primitive recursion,*
lem:prim-rec *it can be defined from f , g , the functions zero, succ, P_i^n , add, mult, $\chi_=$, using composition and regular minimization.*

Proof. First, define an auxiliary function $\hat{h}(\vec{x}, y)$ which returns the least number d such that d codes a sequence which satisfies

1. $(d)_0 = f(\vec{x})$, and
2. for each $i < x$, $(d)_{i+1} = g(\vec{x}, i, (d)_i)$,

where now $(d)_i$ is short for $\beta(d, i)$. In other words, \hat{h} returns the sequence $\langle h(\vec{x}, 0), h(\vec{x}, 1), \dots, h(\vec{x}, y) \rangle$. We can write \hat{h} as

$$\hat{h}(\vec{x}, y) = \mu d (\beta(d, 0) = f(\vec{x}) \wedge (\forall i < y) \beta(d, i + 1) = g(\vec{x}, i, \beta(d, i))).$$

Note: no primitive recursion is needed here, just minimization. The function we minimize is regular because of the beta function lemma ??.

But now we have

$$h(\vec{x}, y) = \beta(\hat{h}(\vec{x}, y), y),$$

so h can be defined from the basic functions using just composition and regular minimization. \square

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Bibliography