

req.1 Regular Minimization is Representable in \mathbf{Q}

inc:req:min:
sec Let's consider unbounded search. Suppose $g(x, z)$ is regular and representable in \mathbf{Q} , say by the formula $\varphi_g(x, z, y)$. Let f be defined by $f(z) = \mu x [g(x, z) = 0]$. We would like to find a formula $\varphi_f(z, y)$ representing f . The value of $f(z)$ is that number x which (a) satisfies $g(x, z) = 0$ and (b) is the least such, i.e., for any $w < x$, $g(w, z) \neq 0$. So the following is a natural choice:

$$\varphi_f(z, y) \equiv \varphi_g(y, z, 0) \wedge \forall w (w < y \rightarrow \neg \varphi_g(w, z, 0)).$$

In the general case, of course, we would have to replace z with z_0, \dots, z_k .

The proof, again, will involve some lemmas about things \mathbf{Q} is strong enough to prove.

inc:req:min:
lem:succ **Lemma req.1.** For every constant symbol a and every natural number n ,

$$\mathbf{Q} \vdash (a' + \bar{n}) = (a + \bar{n})'.$$

Proof. The proof is, as usual, by induction on n . In the base case, $n = 0$, we need to show that \mathbf{Q} proves $(a' + 0) = (a + 0)'$. But we have:

<small>inc:req:min: step1</small>	$\mathbf{Q} \vdash (a' + 0) = a'$	by axiom Q_4	(1)
<small>inc:req:min: step2</small>	$\mathbf{Q} \vdash (a + 0) = a$	by axiom Q_4	(2)
<small>inc:req:min: step3</small>	$\mathbf{Q} \vdash (a + 0)' = a'$	by eq. (2)	(3)
	$\mathbf{Q} \vdash (a' + 0) = (a + 0)'$	by eq. (1) and eq. (3)	

In the induction step, we can assume that we have shown that $\mathbf{Q} \vdash (a' + \bar{n}) = (a + \bar{n})'$. Since $\overline{n+1}$ is \bar{n}' , we need to show that \mathbf{Q} proves $(a' + \bar{n}') = (a + \bar{n}')'$. We have:

<small>inc:req:min: step5</small>	$\mathbf{Q} \vdash (a' + \bar{n}') = (a' + \bar{n})'$	by axiom Q_5	(4)
<small>inc:req:min: step6</small>	$\mathbf{Q} \vdash (a' + \bar{n}') = (a + \bar{n}')'$	inductive hypothesis	(5)
	$\mathbf{Q} \vdash (a' + \bar{n}') = (a + \bar{n}')'$	by eq. (4) and eq. (5).	□

It is again worth mentioning that this is weaker than saying that \mathbf{Q} proves $\forall x \forall y (x' + y) = (x + y)'$. Although this sentence is true in \mathfrak{N} , \mathbf{Q} does not prove it.

inc:req:min:
lem:less-zero **Lemma req.2.** $\mathbf{Q} \vdash \forall x \neg x < 0$.

Proof. We give the proof informally (i.e., only giving hints as to how to construct the formal derivation).

We have to prove $\neg a < 0$ for an arbitrary a . By the definition of $<$, we need to prove $\neg \exists y (y' + a) = 0$ in \mathbf{Q} . We'll assume $\exists y (y' + a) = 0$ and prove a contradiction. Suppose $(b' + a) = 0$. Using Q_3 , we have that $a = 0 \vee \exists y a = y'$. We distinguish cases.

Case 1: $a = 0$ holds. From $(b' + a) = 0$, we have $(b' + 0) = 0$. By axiom Q_4 of \mathbf{Q} , we have $(b' + 0) = b'$, and hence $b' = 0$. But by axiom Q_2 we also have $b' \neq 0$, a contradiction.

Case 2: For some c , $a = c'$. But then we have $(b' + c') = 0$. By axiom Q_5 , we have $(b' + c)' = 0$, again contradicting axiom Q_2 . \square

Lemma req.3. *For every natural number n ,*

*inc:req:min:
lem:less-nsucc*

$$\mathbf{Q} \vdash \forall x (x < \overline{n+1} \rightarrow (x = 0 \vee \dots \vee x = \bar{n})).$$

Proof. We use induction on n . Let us consider the base case, when $n = 0$. In that case, we need to show $a < \bar{1} \rightarrow a = 0$, for arbitrary a . Suppose $a < \bar{1}$. Then by the defining axiom for $<$, we have $\exists y (y' + a) = 0'$ (since $\bar{1} \equiv 0'$).

Suppose b has that property, i.e., we have $(b' + a) = 0'$. We need to show $a = 0$. By axiom Q_3 , we have either $a = 0$ or that there is a c such that $a = c'$. In the former case, there is nothing to show. So suppose $a = c'$. Then we have $(b' + c') = 0'$. By axiom Q_5 of \mathbf{Q} , we have $(b' + c)' = 0'$. By axiom Q_1 , we have $(b' + c) = 0$. But this means, by axiom Q_8 , that $c < 0$, contradicting [Lemma req.2](#).

Now for the inductive step. We prove the case for $n + 1$, assuming the case for n . So suppose $a < \overline{n+2}$. Again using Q_3 we can distinguish two cases: $a = 0$ and for some b , $a = b'$. In the first case, $a = 0 \vee \dots \vee a = \overline{n+1}$ follows trivially. In the second case, we have $c' < \overline{n+2}$, i.e., $c' < \overline{n+1}'$. By axiom Q_8 , for some d , $(d' + c') = \overline{n+1}'$. By axiom Q_5 , $(d' + c)' = \overline{n+1}'$. By axiom Q_1 , $(d' + c) = \overline{n+1}$, and so $c < \overline{n+1}$ by axiom Q_8 . By inductive hypothesis, $c = 0 \vee \dots \vee c = \bar{n}$. From this, we get $c' = 0' \vee \dots \vee c' = \bar{n}'$ by logic, and so $a = \bar{1} \vee \dots \vee a = \overline{n+1}$ since $a = c'$. \square

Lemma req.4. *For every natural number m ,*

*inc:req:min:
lem:trichotomy*

$$\mathbf{Q} \vdash \forall y ((y < \bar{m} \vee \bar{m} < y) \vee y = \bar{m}).$$

Proof. By induction on m . First, consider the case $m = 0$. $\mathbf{Q} \vdash \forall y (y = 0 \vee \exists z (y = z'))$ by Q_3 . Let a be arbitrary. Then either $a = 0$ or for some b , $a = b'$. In the former case, we also have $(a < 0 \vee 0 < a) \vee a = 0$. But if $a = b'$, then $(b' + 0) = (a + 0)$ by the logic of $=$. By Q_4 , $(a + 0) = a$, so we have $(b' + 0) = a$, and hence $\exists z (z' + 0) = a$. By the definition of $<$ in Q_8 , $0 < a$. If $0 < a$, then also $(0 < a \vee a < 0) \vee a = 0$.

Now suppose we have

$$\mathbf{Q} \vdash \forall y ((y < \bar{m} \vee \bar{m} < y) \vee y = \bar{m})$$

and we want to show

$$\mathbf{Q} \vdash \forall y ((y < \overline{m+1} \vee \overline{m+1} < y) \vee y = \overline{m+1})$$

Let a be arbitrary. By Q_3 , either $a = 0$ or for some b , $a = b'$. In the first case, we have $\bar{m}' + a = \overline{m+1}$ by Q_4 , and so $a < \overline{m+1}$ by Q_8 .

Now consider the second case, $a = b'$. By the induction hypothesis, $(b < \overline{m} \vee \overline{m} < b) \vee b = \overline{m}$.

The first disjunct $b < \overline{m}$ is equivalent (by Q_8) to $\exists z (z' + b) = \overline{m}$. Suppose c has this property. If $(c' + b) = \overline{m}$, then also $(c' + b)' = \overline{m}'$. By Q_5 , $(c' + b)' = (c' + b')$. Hence, $(c' + b') = \overline{m}'$. We get $\exists u (u' + b') = \overline{m} + \overline{1}$ by existentially generalizing on c' and keeping in mind that $\overline{m}' \equiv \overline{m} + \overline{1}$. Hence, if $b < \overline{m}$ then $b' < \overline{m} + \overline{1}$ and so $a < \overline{m} + \overline{1}$.

Now suppose $\overline{m} < b$, i.e., $\exists z (z' + \overline{m}) = b$. Suppose c is such a z , i.e., $(c' + \overline{m}) = b$. By logic, $(c' + \overline{m})' = b'$. By Q_5 , $(c' + \overline{m}') = b'$. Since $a = b'$ and $\overline{m}' \equiv \overline{m} + \overline{1}$, $(c' + \overline{m} + \overline{1}) = a$. By Q_8 , $\overline{m} + \overline{1} < a$.

Finally, assume $b = \overline{m}$. Then, by logic, $b' = \overline{m}'$, and so $a = \overline{m} + \overline{1}$.

Hence, from each disjunct of the case for m and b , we can obtain the corresponding disjunct for for $m + 1$ and a . \square

inc:req:min: **Proposition req.5.** *prop:rep-minimization* If $\varphi_g(x, z, y)$ represents $g(x, z)$ in \mathbf{Q} , then

$$\varphi_f(z, y) \equiv \varphi_g(y, z, 0) \wedge \forall w (w < y \rightarrow \neg \varphi_g(w, z, 0))$$

represents $f(z) = \mu x [g(x, z) = 0]$.

Proof. First we show that if $f(n) = m$, then $\mathbf{Q} \vdash \varphi_f(\overline{n}, \overline{m})$, i.e.,

$$\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, 0) \wedge \forall w (w < \overline{m} \rightarrow \neg \varphi_g(w, \overline{n}, 0)).$$

Since $\varphi_g(x, z, y)$ represents $g(x, z)$ and $g(m, n) = 0$ if $f(n) = m$, we have

$$\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, 0).$$

If $f(n) = m$, then for every $k < m$, $g(k, n) \neq 0$. So

$$\mathbf{Q} \vdash \neg \varphi_g(\overline{k}, \overline{n}, 0).$$

We get that

$$\mathbf{Q} \vdash \forall w (w < \overline{m} \rightarrow \neg \varphi_g(w, \overline{n}, 0)). \quad (6)$$

inc:req:min:
rep-less

by **Lemma req.2** in case $m = 0$ and by **Lemma req.3** otherwise.

Now let's show that if $f(n) = m$, then $\mathbf{Q} \vdash \forall y (\varphi_f(\overline{n}, y) \rightarrow y = \overline{m})$. We again sketch the argument informally, leaving the formalization to the reader.

Suppose $\varphi_f(\overline{n}, b)$. From this we get (a) $\varphi_g(b, \overline{n}, 0)$ and (b) $\forall w (w < b \rightarrow \neg \varphi_g(w, \overline{n}, 0))$. By **Lemma req.4**, $(b < \overline{m} \vee \overline{m} < b) \vee b = \overline{m}$. We'll show that both $b < \overline{m}$ and $\overline{m} < b$ leads to a contradiction.

If $\overline{m} < b$, then $\neg \varphi_g(\overline{m}, \overline{n}, 0)$ from (b). But $m = f(n)$, so $g(m, n) = 0$, and so $\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, 0)$ since φ_g represents g . So we have a contradiction.

Now suppose $b < \overline{m}$. Then since $\mathbf{Q} \vdash \forall w (w < \overline{m} \rightarrow \neg \varphi_g(w, \overline{n}, 0))$ by **eq. (6)**, we get $\neg \varphi_g(b, \overline{n}, 0)$. This again contradicts (a). \square

Photo Credits

Bibliography