req.1 Regular Minimization is Representable in Q

Let’s consider unbounded search. Suppose \( g(x, z) \) is regular and representable in \( Q \), say by the formula \( \varphi_g(x, z, y) \). Let \( f \) be defined by \( f(z) = \mu x \ [g(x, z) = 0] \). We would like to find a formula \( \varphi_f(z, y) \) representing \( f \). The value of \( f(z) \) is that number \( x \) which (a) satisfies \( g(x, z) = 0 \) and (b) is the least such, i.e., for any \( w < x \), \( g(w, z) \neq 0 \). So the following is a natural choice:

\[
\varphi_f(z, y) \equiv \varphi_g(y, z, 0) \land \forall w \ (w < y \rightarrow \neg \varphi_g(w, z, 0)).
\]

In the general case, of course, we would have to replace \( z \) with \( z_0, \ldots, z_k \).

Lemma req.1. For every constant symbol \( a \) and every natural number \( n \),

\[ Q \vdash (a' + n) = (a + n'). \]

Proof. The proof is, as usual, by induction on \( n \). In the base case, \( n = 0 \), we need to show that \( Q \) proves \((a + 0) = (a_0)' \). But we have:

\[
\begin{align*}
Q \vdash (a' + 0) &= a' & \text{by axiom } Q_4 \\
Q \vdash (a + 0) &= a & \text{by axiom } Q_4 \\
Q \vdash (a + 0)' &= a' & \text{by eq. (2)} \\
Q \vdash (a' + 0) = (a + 0)' & \text{by eq. (1) and eq. (3)}
\end{align*}
\]

In the induction step, we can assume that we have shown that \( Q \vdash (a' + n) = (a + n') \). Since \( n + 1 \) is \( n' \), we need to show that \( Q \) proves \((a' + n') = (a + n')' \). We have:

\[
\begin{align*}
Q \vdash (a' + n') &= (a' + n')' & \text{by axiom } Q_5 \\
Q \vdash (a' + n') &= (a + n')' & \text{inductive hypothesis} \\
Q \vdash (a' + n')' &= (a + n')' & \text{by eq. (4) and eq. (5)}.
\end{align*}
\]

It is again worth mentioning that this is weaker than saying that \( Q \) proves \( \forall x \forall y (x' + y) = (x + y)' \). Although this sentence is true in \( \mathfrak{N} \), \( Q \) does not prove it.

Lemma req.2. \( Q \vdash \forall x \neg x < 0 \).

Proof. We give the proof informally (i.e., only giving hints as to how to construct the formal derivation).

We have to prove \( \neg a < 0 \) for an arbitrary \( a \). By the definition of \( < \), we need to prove \( \neg \exists y (y' + a) = 0 \) in \( Q \). We’ll assume \( \exists y (y' + a) = 0 \) and prove a contradiction. Suppose \( (b' + a) = 0 \). Using \( Q_3 \), we have that \( a = 0 \lor \exists y a = y' \).

We distinguish cases.
Case 1: \( a = 0 \) holds. From \((b' + a) = a\), we have \((b' + 0) = 0\). By axiom \(Q_4\) of \(Q\), we have \((b' + 0) = b'\), and hence \(b' = 0\). But by axiom \(Q_2\) we also have \(b' \neq 0\), a contradiction.

Case 2: For some \(c\), \(a = c'\). But then we have \((b' + c') = 0\). By axiom \(Q_5\), we have \((b' + c')' = 0\), again contradicting axiom \(Q_2\).

\[\square\]

**Lemma req.3.** For every natural number \(n\),

\[Q \vdash \forall x (x < n + 1 \to (x = 0 \lor \cdots \lor x = m)).\]

**Proof.** We use induction on \(n\). Let us consider the base case, when \(n = 0\). In that case, we need to show \(a < 1 \to a = 0\), for arbitrary \(a\). Suppose \(a < 1\). Then by the defining axiom for \(<\), we have \(\exists y (y' + a = 0')\) (since \(1 \equiv 0'\)).

Suppose \(b\) has that property, i.e., we have \((b' + a) = 0'\). We need to show \(a = 0\). By axiom \(Q_3\), we have either \(a = 0\) or that there is a \(c\) such that \(a = c'\). In the former case, there is nothing to show. So suppose \(a = c'\). Then we have \((b' + c') = 0'\). By axiom \(Q_5\) of \(Q\), we have \((b' + c')' = 0'\). By axiom \(Q_1\), we have \((b' + c) = 0\). But this means, by axiom \(Q_8\), that \(c < 0\), contradicting Lemma req.2.

Now for the inductive step. We prove the case for \(n + 1\), assuming the case for \(n\). So suppose \(a < n + 2\). Again using \(Q_3\) we can distinguish two cases: \(a = 0\) and for some \(b\), \(a = c'\). In the first case, \(a = 0 \lor \cdots \lor a = n + 1\) follows trivially. In the second case, we have \(c' < n + 2\), i.e., \(c' < n + 1\). By axiom \(Q_8\), for some \(d\), \((d' + c') = n + 1\). By axiom \(Q_5\), \((d' + c')' = n + 1\). By axiom \(Q_1\), \((d' + c) = n + 1\), and so \(c < n + 1\) by axiom \(Q_8\). By inductive hypothesis, \(c = 0 \lor \cdots \lor c = \pi\). From this, we get \(c' = a' \lor \cdots \lor c' = \pi'\) by logic, and so \(a = 1 \lor \cdots \lor a = n + 1\) since \(a = c'\).

\[\square\]

**Lemma req.4.** For every natural number \(m\),

\[Q \vdash \forall y ((y < m \lor m < y) \lor y = m).\]

**Proof.** By induction on \(m\). First, consider the case \(m = 0\). \(Q \vdash \forall y (y = 0 \lor \exists z y' = z')\) by \(Q_3\). Let \(a\) be arbitrary. Then either \(a = 0\) or for some \(b\), \(a = b'\). In the former case, we also have \((a < 0 \lor 0 < a) \lor a = 0\). But if \(a = b'\), then \((b' + a) = (a + 0)\) by the logic of \(\equiv\). By \(Q_4\), \((a + 0) = a\), so we have \((b' + a) = a\), and hence \(\exists z (z' + a) = a\). By the definition of \(<\) in \(Q_8\), \(0 < a\). If \(0 < a\), then also \((0 < a \lor a < 0) \lor a = 0\).

Now suppose we have

\[Q \vdash \forall y ((y < m \lor m < y) \lor y = m)\]

and we want to show

\[Q \vdash \forall y ((y < m + 1 \lor m + 1 < y) \lor y = m + 1)\]

Let \(a\) be arbitrary. By \(Q_3\), either \(a = 0\) or for some \(b\), \(a = b'\). In the first case, we have \(m' + a = m + 1\) by \(Q_4\), and so \(a < m + 1\) by \(Q_8\).
Now consider the second case, $a = b'$. By the induction hypothesis, $(b < m \lor m < b) \lor b = m$.

The first disjunct $b < m$ is equivalent (by $Q_8$) to $\exists z (z' + b) = m$. Suppose $c$ has this property. If $(c' + b) = m$, then also $(c' + b') = m'$. By $Q_5$, $(c' + b)' = (c' + b')$. Hence, $(c' + b') = m'$. We get $\exists u (u' + b') = m + 1$ by existentially generalizing on $c'$ and keeping in mind that $m' = m + 1$. Hence, if $b < m$ then $b' < m + 1$ and so $a < m + 1$.

Now suppose $m < b$, i.e., $\exists z (z' + m) = b$. Suppose $c$ is such a $z$, i.e., $(c' + m) = b$. By logic, $(c' + m)' = b'$. By $Q_5$, $(c' + m') = b'$. Since $a = b'$ and $m' = m + 1$, $(c' + m + 1) = a$. By $Q_8$, $m + 1 < a$.

Finally, assume $b = m$. Then, by logic, $b' = m'$, and so $a = m + 1$.

Hence, from each disjoint of the case for $m$ and $b$, we can obtain the corresponding disjunct for for $m + 1$ and $a$.

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**Proposition req.5.** If $\varphi_g(x, z, y)$ represents $g(x, z)$ in $Q$, then  

$$ \varphi_f(z, y) \equiv \varphi_g(y, z, o) \land \forall w (w < y \rightarrow \neg \varphi_g(w, z, o)) $$  

represents $f(z) = \mu x [g(x, z) = 0]$.

**Proof.** First we show that if $f(n) = m$, then $Q \vdash \varphi_f(n, m)$, i.e.,

$$ Q \vdash \varphi_g(m, n, o) \land \forall w (w < m \rightarrow \neg \varphi_g(w, n, o)). $$

Since $\varphi_g(x, z, y)$ represents $g(x, z)$ and $g(m, n) = 0$ if $f(n) = m$, we have

$$ Q \vdash \varphi_g(m, n, o). $$

If $f(n) = m$, then for every $k < m$, $g(k, n) \neq 0$. So

$$ Q \vdash \neg \varphi_g(k, n, o). $$

We get that

$$ Q \vdash \forall w (w < m \rightarrow \neg \varphi_g(w, n, o)). $$ (6)

by Lemma req.2 in case $m = 0$ and by Lemma req.3 otherwise.

Now let’s show that if $f(n) = m$, then $Q \vdash \forall y (\varphi_f(y, y) \rightarrow y = m)$. We again sketch the argument informally, leaving the formalization to the reader.

Suppose $\varphi_f(n, b)$. From this we get (a) $\varphi_g(b, n, o)$ and (b) $\forall w (w < b \rightarrow \neg \varphi_g(w, n, o))$. By Lemma req.4, $(b < m \lor m < b) \lor b = m$. We’ll show that both $b < m$ and $m < b$ leads to a contradiction.

If $m < b$, then $\neg \varphi_g(m, n, o)$ from (b). But $m = f(n)$, so $g(m, n) = 0$, and so $Q \vdash \varphi_g(m, n, o)$ since $\varphi_g$ represents $g$. So we have a contradiction.

Now suppose $b < m$. Then since $Q \vdash \forall w (w < m \rightarrow \neg \varphi_g(w, n, o))$ by eq. (6), we get $\neg \varphi_g(b, n, o)$. This again contradicts (a).
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Bibliography