

req.1 Regular Minimization is Representable in \mathbf{Q}

inc:req:min:
sec Let's consider unbounded search. Suppose $g(x, z)$ is regular and representable in \mathbf{Q} , say by the formula $\varphi_g(x, z, y)$. Let f be defined by $f(z) = \mu x [g(x, z) = 0]$. We would like to find a formula $\varphi_f(z, y)$ representing f . The value of $f(z)$ is that number x which (a) satisfies $g(x, z) = 0$ and (b) is the least such, i.e., for any $w < x$, $g(w, z) \neq 0$. So the following is a natural choice:

$$\varphi_f(z, y) \equiv \varphi_g(y, z, 0) \wedge \forall w (w < y \rightarrow \neg \varphi_g(w, z, 0)).$$

In the general case, of course, we would have to replace z with z_0, \dots, z_k .

The proof, again, will involve some lemmas about things \mathbf{Q} is strong enough to prove.

inc:req:min:
lem:succ **Lemma req.1.** *For every variable x and every natural number n ,*

$$\mathbf{Q} \vdash (x' + \bar{n}) = (x + \bar{n})'.$$

Proof. The proof is, as usual, by induction on n . In the base case, $n = 0$, we need to show that \mathbf{Q} proves $(x' + 0) = (x + 0)'$. But we have:

| | | | |
|---------------------------------------|--|----------------|-----|
| <small>inc:req:min:</small> | $\mathbf{Q} \vdash (x' + 0) = x'$ | by axiom Q_4 | (1) |
| <small>inc:req:mfn: step1</small> | $\mathbf{Q} \vdash (x + 0) = x$ | by axiom Q_4 | (2) |
| <small>inc:req:mfn: step2</small> | $\mathbf{Q} \vdash (x + 0)' = x'$ | by eq. (2) | (3) |
| <small>step3</small> | $\mathbf{Q} \vdash (x' + 0) = (x + 0)'$ by eq. (1) and eq. (3) | | |

In the induction step, we can assume that we have shown that $\mathbf{Q} \vdash (x' + \bar{n}) = (x + \bar{n})'$. Since $\bar{n} + \bar{1}$ is \bar{n}' , we need to show that \mathbf{Q} proves $(x' + \bar{n}') = (x + \bar{n}')'$. We have:

| | | | |
|---------------------------------------|---|----------------------|-----|
| <small>inc:req:min:</small> | $\mathbf{Q} \vdash (x' + \bar{n}') = (x' + \bar{n})'$ | by axiom Q_5 | (4) |
| <small>inc:req:mfn: step5</small> | $\mathbf{Q} \vdash (x' + \bar{n}') = (x + \bar{n}')'$ | inductive hypothesis | (5) |
| <small>step6</small> | $\mathbf{Q} \vdash (x' + \bar{n}') = (x + \bar{n}')'$ by eq. (4) and eq. (5). | | |

□

It is again worth mentioning that this is weaker than saying that \mathbf{Q} proves $\forall x \forall y (x' + y) = (x + y)'$. Although this sentence is true in \mathfrak{N} , \mathbf{Q} does not prove it.

inc:req:min:
lem:less **Lemma req.2.**

1. $\mathbf{Q} \vdash \forall x \neg x < 0$.
2. For every natural number n ,

$$\mathbf{Q} \vdash \forall x (x < \overline{n+1} \rightarrow (x = 0 \vee \dots \vee x = \bar{n})).$$

Proof. Let us do 1 and part of 2, informally (i.e., only giving hints as to how to construct the formal derivation).

For part 1, by the definition of $<$, we need to prove $\neg\exists y(y' + x) = 0$ in \mathbf{Q} , which is equivalent (using the axioms and rules of first-order logic) to $\forall y(y' + x) \neq 0$. Here is the idea: suppose $(y' + x) = 0$. If $x = 0$, we have $(y' + 0) = 0$. But by axiom Q_4 of \mathbf{Q} , we have $(y' + 0) = y'$, and by axiom Q_2 we have $y' \neq 0$, a contradiction. So $\forall y(y' + x) \neq 0$. If $x \neq 0$, by axiom Q_3 , there is a z such that $x = z'$. But then we have $(y' + z') = 0$. By axiom Q_5 , we have $(y' + z)' = 0$, again contradicting axiom Q_2 .

For part 2, use induction on n . Let us consider the base case, when $n = 0$. In that case, we need to show $x < \bar{1} \rightarrow x = 0$. Suppose $x < \bar{1}$. Then by the defining axiom for $<$, we have $\exists y(y' + x) = 0'$. Suppose y has that property; so we have $y' + x = 0'$.

We need to show $x = 0$. By axiom Q_3 , if $x \neq 0$, we get $x = z'$ for some z . Then we have $(y' + z') = 0'$. By axiom Q_5 of \mathbf{Q} , we have $(y' + z)' = 0'$. By axiom Q_1 , we have $(y' + z) = 0$. But this means, by definition, $z < 0$, contradicting part 1. \square

Lemma req.3. For every $m \in \mathbb{N}$,

$$\mathbf{Q} \vdash \forall y ((y < \bar{m} \vee \bar{m} < y) \vee y = \bar{m}).$$

*inc:req:min:
lem:trichotomy*

Proof. By induction on m . First, consider the case $m = 0$. $\mathbf{Q} \vdash \forall y (y \neq 0 \rightarrow \exists z y = z')$ by Q_3 . But if $y = z'$, then $(z' + 0) = (y + 0)$ by the logic of $=$. By Q_4 , $(y + 0) = y$, so we have $(z' + 0) = y$, and hence $\exists z (z' + 0) = y$. By the definition of $<$ in Q_8 , $0 < y$. If $0 < y$, then also $0 < y \vee y < 0$. We obtain: $y \neq 0 \rightarrow (0 < y \vee y < 0)$, which is equivalent to $(0 < y \vee y < 0) \vee y = 0$.

Now suppose we have

$$\mathbf{Q} \vdash \forall y ((y < \bar{m} \vee \bar{m} < y) \vee y = \bar{m})$$

and we want to show

$$\mathbf{Q} \vdash \forall y ((y < \overline{m+1} \vee \overline{m+1} < y) \vee y = \overline{m+1})$$

The first disjunct $y < \bar{m}$ is equivalent (by Q_8) to $\exists z (z' + y) = \bar{m}$. If $(z' + y) = \bar{m}$, then also $(z' + y)' = \bar{m}'$. By Q_4 , $(z' + y)' = (z'' + y)$. Hence, $(z'' + y) = \bar{m}'$. We get $\exists u (u' + y) = \overline{m+1}$ by existentially generalizing on z' and keeping in mind that \bar{m}' is $\overline{m+1}$. Hence, if $y < \bar{m}$ then $y < \overline{m+1}$.

Now suppose $\bar{m} < y$, i.e., $\exists z (z' + \bar{m}) = y$. By Q_3 and some logic, we have $z = 0 \vee \exists u z = u'$. If $z = 0$, we have $(0' + \bar{m}) = y$. Since $\mathbf{Q} \vdash (0' + \bar{m}) = \overline{m+1}$, we have $y = \overline{m+1}$. Now suppose $\exists u z = u'$. Then:

$$\begin{aligned} y &= (z' + \bar{m}) && \text{by assumption} \\ (z' + \bar{m}) &= (u'' + \bar{m}) && \text{from } z = u' \\ (u'' + \bar{m}) &= (u' + \bar{m})' && \text{by Lemma req.1} \\ (u' + \bar{m})' &= (u' + \bar{m}') && \text{by } Q_5, \text{ so} \\ y &= (u' + \overline{m+1}) \end{aligned}$$

By existential generalization, $\exists u (u' + \overline{m+1}) = y$, i.e., $\overline{m+1} < y$. So, if $\overline{m} < y$, then $\overline{m+1} < y \vee y = \overline{m+1}$.

Finally, assume $y = \overline{m}$. Then, since $\mathbf{Q} \vdash (o' + \overline{m}) = \overline{m+1}$, $(o' + y) = \overline{m+1}$. From this we get $\exists z (z' + y) = \overline{m+1}$, or $y < \overline{m+1}$.

Hence, from each disjunct of the case for m , we can obtain the case for $m+1$. \square

*inc:req:min:
prop:rep-minimization*

Proposition req.4. *If $\varphi_g(x, z, y)$ represents $g(x, y)$ in \mathbf{Q} , then*

$$\varphi_f(z, y) \equiv \varphi_g(y, z, 0) \wedge \forall w (w < y \rightarrow \neg \varphi_g(w, z, 0)).$$

represents $f(z) = \mu x [g(x, z) = 0]$.

Proof. First we show that if $f(n) = m$, then $\mathbf{Q} \vdash \varphi_f(\overline{n}, \overline{m})$, i.e.,

$$\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, 0) \wedge \forall w (w < \overline{m} \rightarrow \neg \varphi_g(w, \overline{n}, 0)).$$

Since $\varphi_g(x, z, y)$ represents $g(x, z)$ and $g(m, n) = 0$ if $f(n) = m$, we have

$$\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, 0).$$

If $f(n) = m$, then for every $k < m$, $g(k, n) \neq 0$. So

$$\mathbf{Q} \vdash \neg \varphi_g(\overline{k}, \overline{n}, 0).$$

We get that

$$\mathbf{Q} \vdash \forall w (w < \overline{m} \rightarrow \neg \varphi_g(w, \overline{n}, 0)). \quad (6)$$

*inc:req:min:
rep-less*

by Lemma req.2 (by (1) in case $m = 0$ and by (2) otherwise).

Now let's show that if $f(n) = m$, then $\mathbf{Q} \vdash \forall y (\varphi_f(\overline{n}, y) \rightarrow y = \overline{m})$. We again sketch the argument informally, leaving the formalization to the reader.

Suppose $\varphi_f(\overline{n}, y)$. From this we get (a) $\varphi_g(y, \overline{n}, 0)$ and (b) $\forall w (w < y \rightarrow \neg \varphi_g(w, \overline{n}, 0))$. By Lemma req.3, $(y < \overline{m} \vee \overline{m} < y) \vee y = \overline{m}$. We'll show that both $y < \overline{m}$ and $\overline{m} < y$ leads to a contradiction.

If $\overline{m} < y$, then $\neg \varphi_g(\overline{m}, \overline{n}, 0)$ from (b). But $m = f(n)$, so $g(m, n) = 0$, and so $\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, 0)$ since φ_g represents g . So we have a contradiction.

Now suppose $y < \overline{m}$. Then since $\mathbf{Q} \vdash \forall w (w < \overline{m} \rightarrow \neg \varphi_g(w, \overline{n}, 0))$ by eq. (6), we get $\neg \varphi_g(y, \overline{n}, 0)$. This again contradicts (a). \square

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Bibliography