Let's consider unbounded search. Suppose \( g(x, z) \) is regular and representable in \( Q \), say by the formula \( \varphi_g(x, z, y) \). We would like to find a formula \( \varphi_f(z, y) \) representing \( f \). The value of \( f(z) \) is that number \( x \) which (a) satisfies \( g(x, z) = 0 \) and (b) is the least such, i.e., for any \( w < x \), \( g(w, z) \neq 0 \). So the following is a natural choice:

\[
\varphi_f(z, y) \equiv \varphi_g(y, z, 0) \land \forall w (w < y \rightarrow \neg \varphi_g(w, z, 0)).
\]

In the general case, of course, we would have to replace \( z \) with \( z_0, \ldots, z_k \).

**Lemma req.1.** For every constant symbol \( a \) and every natural number \( n \),

\[
Q \vdash (a' + \pi) = (a + \pi)'.
\]

**Proof.** The proof is, as usual, by induction on \( n \). In the base case, \( n = 0 \), we need to show that \( Q \) proves \( (a' + \pi) = (a + \pi)' \). But we have:

\[
\begin{align*}
Q & \vdash (a' + \pi) = a' \quad \text{by axiom } Q_4 \quad (1) \\
Q & \vdash (a + \pi) = a \quad \text{by axiom } Q_4 \quad (2) \\
Q & \vdash (a + \pi)' = a' \quad \text{by eq. (2)} \quad (3) \\
Q & \vdash (a' + \pi) = (a + \pi)' \quad \text{by eq. (1) and eq. (3)} \quad (4)
\end{align*}
\]

In the induction step, we can assume that we have shown that \( Q \vdash (a' + \pi) = (a + \pi)' \). Since \( n + 1 \) is \( \pi' \), we need to show that \( Q \) proves \( (a' + \pi') = (a + \pi')' \). We have:

\[
\begin{align*}
Q & \vdash (a' + \pi') = (a' + \pi)' \quad \text{by axiom } Q_5 \quad (4) \\
Q & \vdash (a' + \pi') = (a + \pi')' \quad \text{inductive hypothesis} \quad (5) \\
Q & \vdash (a' + \pi)' = (a + \pi')' \quad \text{by eq. (4) and eq. (5)}. \quad \square
\end{align*}
\]

It is again worth mentioning that this is weaker than saying that \( Q \) proves \( \forall x \forall y (x' + y) = (x + y)' \). Although this sentence is true in \( \mathfrak{N} \), \( Q \) does not prove it.

**Lemma req.2.** \( Q \vdash \forall x \neg x < 0 \).

**Proof.** We give the proof informally (i.e., only giving hints as to how to construct the formal derivation).

We have to prove \( \neg a < 0 \) for an arbitrary \( a \). By the definition of \( < \), we need to prove \( \neg \exists y (y' + a) = 0 \) in \( Q \). We'll assume \( \exists y (y' + a) = 0 \) and prove a contradiction. Suppose \( (b' + a) = 0 \). Using \( Q_3 \), we have that \( a = 0 \lor \exists y a = y' \). We distinguish cases.
Case 1: \( a = \mathfrak{0} \) holds. From \( (b' + a) = \mathfrak{0} \), we have \( (b' + \mathfrak{0}) = \mathfrak{0} \). By axiom \( Q_4 \) of \( Q \), we have \( (b' + \mathfrak{0}) = b' \), and hence \( b' = \mathfrak{0} \). But by axiom \( Q_2 \) we also have \( b' \neq \mathfrak{0} \), a contradiction.

Case 2: For some \( c, a = c' \). But then we have \( (b' + c') = \mathfrak{0} \). By axiom \( Q_5 \), we have \( (b' + c')' = \mathfrak{0} \), again contradicting axiom \( Q_2 \).

**Lemma req.3.** For every natural number \( n \),

\[
Q \vdash \forall x (x < n + 1 \rightarrow (x = \mathfrak{0} \lor \cdots \lor x = \mathfrak{m})).
\]

**Proof.** We use induction on \( n \). Let us consider the base case, when \( n = 0 \). In that case, we need to show \( a < 1 \rightarrow a = a \), for arbitrary \( a \). Suppose \( a < 1 \).

Then by the defining axiom for \( < \), we have \( \exists y (y' + a) = a' \) (since \( 1 \equiv a' \)).

Suppose \( b \) has that property, i.e., we have \( (b' + a) = a' \). We need to show \( a' \mathfrak{0} \). By axiom \( Q_3 \), we have either \( a = \mathfrak{0} \) or that there is a \( c \) such that \( a = c' \).

In the former case, there is nothing to show. So suppose \( a = c' \). Then we have \( (b' + c') = a' \). By axiom \( Q_5 \) of \( Q \), we have \( (b' + c')' = a' \). By axiom \( Q_1 \), we have \( (b' + c) = \mathfrak{0} \). But this means, by axiom \( Q_8 \), that \( c < \mathfrak{0} \), contradicting Lemma req.2.

Now for the inductive step. We prove the case for \( n + 1 \), assuming the case for \( n \). So suppose \( a < n + \frac{3}{2} \). Again using \( Q_3 \) we can distinguish two cases: \( a = \mathfrak{0} \) and for some \( b, a = c' \). In the first case, \( a = \mathfrak{0} \lor \cdots \lor a = n + 1 \) follows trivially. In the second case, we have \( c' < n + \frac{3}{2} \), i.e., \( c < n + 1 \). By axiom \( Q_8 \), for some \( d, (d' + c') = n + 1 \). By axiom \( Q_5 \), \( (d' + c')' = n + 1 \). By axiom \( Q_1 \), \( (d' + c) = n + 1 \), and so \( c < n + 1 \) by axiom \( Q_8 \). By inductive hypothesis, \( c = \mathfrak{0} \lor \cdots \lor c = \mathfrak{m} \). From this, we get \( c' = a' \lor \cdots \lor c' = \mathfrak{m} \) by logic, and so \( a = \mathfrak{1} \lor \cdots \lor a = n + 1 \) since \( a = c' \).

**Lemma req.4.** For every \( m \in \mathbb{N} \),

\[
Q \vdash \forall y ((y < \overline{m} \lor \overline{m} < y) \lor y = \overline{m}).
\]

**Proof.** By induction on \( m \). First, consider the case \( m = 0 \). \( Q \vdash \forall y (y = \mathfrak{0} \lor \exists z (z' = y)) \) by \( Q_3 \). Let \( a \) be arbitrary. Then either \( a = \mathfrak{0} \) or for some \( b, a = b' \). In the former case, we also have \( (a < \mathfrak{0} \lor \mathfrak{0} < a) \lor a = \mathfrak{0} \). But if \( a = b' \), then \( (b' + a) = (a' + \mathfrak{0}) \) by the logic of \( \mathfrak{0} \). By axiom \( Q_4 \), \( (a' + \mathfrak{0}) = a \), so we have (b' + a) = a. By the definition of \( < \) in \( Q_8 \), \( a < a \). If \( a < a \), then also \( (a < a \lor a < \mathfrak{0}) \lor a = \mathfrak{0} \).

Now suppose we have \( Q \vdash \forall y ((y < \mathfrak{m} \lor \mathfrak{m} < y) \lor y = \mathfrak{m}) \)

and we want to show \( Q \vdash \forall y ((y < \overline{m} \lor \overline{m} < y) \lor y = \overline{m}) \)

Let \( a \) be arbitrary. By \( Q_3 \), either \( a = \mathfrak{0} \) or for some \( b, a = b' \). In the first case, we have \( \overline{m} = a = \overline{m} + 1 \) by \( Q_4 \), and so \( a < \overline{m} \) by \( Q_8 \).
Now consider the second case, \( a = b' \). By the induction hypothesis, \( b < m \lor m < b \lor b = m \).

The first disjunct \( b < m \) is equivalent (by \( Q_7 \)) to \( \exists z (z' + b) = m \). Suppose \( c \) has this property. If \( (c' + b) = m \), then also \( (c' + b)' = m' \). By \( Q_5 \), \( (c' + b)' = (c' + b') \). Hence, \( (c' + b') = m' \). We get \( \exists u (u' + b') = m + 1 \) by existentially generalizing on \( c' \) and keeping in mind that \( m' = m + 1 \). Hence, if \( b < m \) then \( b' = m + 1 \) and so \( a = m + 1 \).

Now suppose \( m < b \), i.e., \( \exists z (z' + m) = b \). Suppose \( c \) is such a \( z \), i.e., \( (c' + m) = m \). By logic, \( (c' + m)' = b' \). By \( Q_5 \), \( (c' + m') = b' \). Since \( a = b' \) and \( m' = m + 1 \), \( (c' + m + 1) = a \). By \( Q_8 \), \( m + 1 < a \).

Finally, assume \( b = m \). Then, by logic, \( b' = m' \), and so \( a = m + 1 \).

Hence, from each disjunct for the case \( m \) and \( b \), we can obtain the corresponding disjunct for \( m + 1 \) and \( a \).

**Proposition req.5.** If \( \varphi_g(x, z, y) \) represents \( g(x, y) \) in \( Q \), then

\[
\varphi_f(z, y) \equiv \varphi_g(y, z, o) \land \forall w (w < y \rightarrow \neg \varphi_g(w, z, o)).
\]

represents \( f(z) = \mu x [g(x, z) = 0] \).

**Proof.** First we show that if \( f(n) = m \), then \( Q \vdash \varphi_f(n, m) \), i.e.,

\[
Q \vdash \varphi_g(m, n, o) \land \forall w (w < m \rightarrow \neg \varphi_g(w, n, o)).
\]

Since \( \varphi_g(x, z, y) \) represents \( g(x, z) \) and \( g(m, n) = 0 \) if \( f(n) = m \), we have

\[
Q \vdash \varphi_g(m, n, o).
\]

If \( f(n) = m \), then for every \( k < m \), \( g(k, n) \neq 0 \). So

\[
Q \vdash \neg \varphi_g(k, n, o).
\]

We get that

\[
Q \vdash \forall w (w < m \rightarrow \neg \varphi_g(w, n, o)).
\]

by **Lemma req.2** in case \( m = 0 \) and by **Lemma req.3** otherwise.

Now let’s show that if \( f(n) = m \), then \( Q \vdash \forall y (\varphi_f(n, y) \rightarrow y = m) \). We again sketch the argument informally, leaving the formalization to the reader.

Suppose \( \varphi_f(n, b) \). From this we get (a) \( \varphi_g(b, n, o) \) and (b) \( \forall w (w < b \rightarrow \neg \varphi_g(w, n, o)) \). By **Lemma req.4**, \( b < m \lor m < b \lor b = m \). We’ll show that both \( b < m \) and \( m < b \) leads to a contradiction.

If \( m < b \), then \( \neg \varphi_g(m, n, o) \) from (b). But \( m = f(n) \), so \( g(m, n) = 0 \), and so \( Q \vdash \varphi_g(m, n, o) \) since \( \varphi_g \) represents \( g \). So we have a contradiction.

Now suppose \( b < m \). Then since \( Q \vdash \forall w (w < m \rightarrow \neg \varphi_g(w, n, o)) \) by eq. (6), we get \( \neg \varphi_g(b, n, o) \). This again contradicts (a).
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Bibliography