

## req.1 Regular Minimization is Representable in $\mathbf{Q}$

inc:req:min:  
sec Let's consider unbounded search. Suppose  $g(x, z)$  is regular and representable in  $\mathbf{Q}$ , say by the formula  $\varphi_g(x, z, y)$ . Let  $f$  be defined by  $f(z) = \mu x [g(x, z) = 0]$ . We would like to find a formula  $\varphi_f(z, y)$  representing  $f$ . The value of  $f(z)$  is that number  $x$  which (a) satisfies  $g(x, z) = 0$  and (b) is the least such, i.e., for any  $w < x$ ,  $g(w, z) \neq 0$ . So the following is a natural choice:

$$\varphi_f(z, y) \equiv \varphi_g(y, z, 0) \wedge \forall w (w < y \rightarrow \neg \varphi_g(w, z, 0)).$$

In the general case, of course, we would have to replace  $z$  with  $z_0, \dots, z_k$ .

The proof, again, will involve some lemmas about things  $\mathbf{Q}$  is strong enough to prove.

inc:req:min:  
lem:succ **Lemma req.1.** *For every variable  $x$  and every natural number  $n$ ,*

$$\mathbf{Q} \vdash (x' + \bar{n}) = (x + \bar{n})'.$$

*Proof.* The proof is, as usual, by induction on  $n$ . In the base case,  $n = 0$ , we need to show that  $\mathbf{Q}$  proves  $(x' + 0) = (x + 0)'$ . But we have:

<small>inc:req:min:</small>	$\mathbf{Q} \vdash (x' + 0) = x'$	by axiom $Q_4$	(1)
<small>inc:req:mfn: step1</small>	$\mathbf{Q} \vdash (x + 0) = x$	by axiom $Q_4$	(2)
<small>inc:req:mfn: step2</small>	$\mathbf{Q} \vdash (x + 0)' = x'$	by eq. (2)	(3)
<small>step3</small>	$\mathbf{Q} \vdash (x' + 0) = (x + 0)'$ by eq. (1) and eq. (3)		

In the induction step, we can assume that we have shown that  $\mathbf{Q} \vdash (x' + \bar{n}) = (x + \bar{n})'$ . Since  $\bar{n} + \bar{1}$  is  $\bar{n}'$ , we need to show that  $\mathbf{Q}$  proves  $(x' + \bar{n}') = (x + \bar{n}')'$ . We have:

<small>inc:req:min:</small>	$\mathbf{Q} \vdash (x' + \bar{n}') = (x' + \bar{n})'$	by axiom $Q_5$	(4)
<small>inc:req:mfn: step5</small>	$\mathbf{Q} \vdash (x' + \bar{n}') = (x + \bar{n}')'$	inductive hypothesis	(5)
<small>step6</small>	$\mathbf{Q} \vdash (x' + \bar{n}') = (x + \bar{n}')'$ by eq. (4) and eq. (5).		

□

It is again worth mentioning that this is weaker than saying that  $\mathbf{Q}$  proves  $\forall x \forall y (x' + y) = (x + y)'$ . Although this sentence is true in  $\mathfrak{N}$ ,  $\mathbf{Q}$  does not prove it.

inc:req:min:  
lem:less **Lemma req.2.**

1.  $\mathbf{Q} \vdash \forall x \neg x < 0$ .
2. For every natural number  $n$ ,

$$\mathbf{Q} \vdash \forall x (x < \overline{n+1} \rightarrow (x = 0 \vee \dots \vee x = \bar{n})).$$

*Proof.* Let us do 1 and part of 2, informally (i.e., only giving hints as to how to construct the formal derivation).

For part 1, by the definition of  $<$ , we need to prove  $\neg\exists y(y' + x) = 0$  in  $\mathbf{Q}$ , which is equivalent (using the axioms and rules of first-order logic) to  $\forall y(y' + x) \neq 0$ . Here is the idea: suppose  $(y' + x) = 0$ . If  $x = 0$ , we have  $(y' + 0) = 0$ . But by axiom  $Q_4$  of  $\mathbf{Q}$ , we have  $(y' + 0) = y'$ , and by axiom  $Q_2$  we have  $y' \neq 0$ , a contradiction. So  $\forall y(y' + x) \neq 0$ . If  $x \neq 0$ , by axiom  $Q_3$ , there is a  $z$  such that  $x = z'$ . But then we have  $(y' + z') = 0$ . By axiom  $Q_5$ , we have  $(y' + z)' = 0$ , again contradicting axiom  $Q_2$ .

For part 2, use induction on  $n$ . Let us consider the base case, when  $n = 0$ . In that case, we need to show  $x < \bar{1} \rightarrow x = 0$ . Suppose  $x < \bar{1}$ . Then by the defining axiom for  $<$ , we have  $\exists y(y' + x) = 0'$ . Suppose  $y$  has that property; so we have  $y' + x = 0'$ .

We need to show  $x = 0$ . By axiom  $Q_3$ , if  $x \neq 0$ , we get  $x = z'$  for some  $z$ . Then we have  $(y' + z') = 0'$ . By axiom  $Q_5$  of  $\mathbf{Q}$ , we have  $(y' + z)' = 0'$ . By axiom  $Q_1$ , we have  $(y' + z) = 0$ . But this means, by definition,  $z < 0$ , contradicting part 1.  $\square$

**Lemma req.3.** For every  $m \in \mathbb{N}$ ,

$$\mathbf{Q} \vdash \forall y ((y < \bar{m} \vee \bar{m} < y) \vee y = \bar{m}).$$

*inc:req:min:  
lem:trichotomy*

*Proof.* By induction on  $m$ . First, consider the case  $m = 0$ .  $\mathbf{Q} \vdash \forall y (y \neq 0 \rightarrow \exists z y = z')$  by  $Q_3$ . But if  $y = z'$ , then  $(z' + 0) = (y + 0)$  by the logic of  $=$ . By  $Q_4$ ,  $(y + 0) = y$ , so we have  $(z' + 0) = y$ , and hence  $\exists z (z' + 0) = y$ . By the definition of  $<$  in  $Q_8$ ,  $0 < y$ . If  $0 < y$ , then also  $0 < y \vee y < 0$ . We obtain:  $y \neq 0 \rightarrow (0 < y \vee y < 0)$ , which is equivalent to  $(0 < y \vee y < 0) \vee y = 0$ .

Now suppose we have

$$\mathbf{Q} \vdash \forall y ((y < \bar{m} \vee \bar{m} < y) \vee y = \bar{m})$$

and we want to show

$$\mathbf{Q} \vdash \forall y ((y < \overline{m+1} \vee \overline{m+1} < y) \vee y = \overline{m+1})$$

The first disjunct  $y < \bar{m}$  is equivalent (by  $Q_8$ ) to  $\exists z (z' + y) = \bar{m}$ . If  $(z' + y) = \bar{m}$ , then also  $(z' + y)' = \bar{m}'$ . By  $Q_4$ ,  $(z' + y)' = (z'' + y)$ . Hence,  $(z'' + y) = \bar{m}'$ . We get  $\exists u (u' + y) = \overline{m+1}$  by existentially generalizing on  $z'$  and keeping in mind that  $\bar{m}'$  is  $\overline{m+1}$ . Hence, if  $y < \bar{m}$  then  $y < \overline{m+1}$ .

Now suppose  $\bar{m} < y$ , i.e.,  $\exists z (z' + \bar{m}) = y$ . By  $Q_3$  and some logic, we have  $z = 0 \vee \exists u z = u'$ . If  $z = 0$ , we have  $(0' + \bar{m}) = y$ . Since  $\mathbf{Q} \vdash (0' + \bar{m}) = \overline{m+1}$ , we have  $y = \overline{m+1}$ . Now suppose  $\exists u z = u'$ . Then:

$$\begin{aligned} y &= (z' + \bar{m}) && \text{by assumption} \\ (z' + \bar{m}) &= (u'' + \bar{m}) && \text{from } z = u' \\ (u'' + \bar{m}) &= (u' + \bar{m})' && \text{by Lemma req.1} \\ (u' + \bar{m})' &= (u' + \bar{m}') && \text{by } Q_5, \text{ so} \\ y &= (u' + \overline{m+1}) \end{aligned}$$

By existential generalization,  $\exists u (u' + \overline{m+1}) = y$ , i.e.,  $\overline{m+1} < y$ . So, if  $\overline{m} < y$ , then  $\overline{m+1} < y \vee y = \overline{m+1}$ .

Finally, assume  $y = \overline{m}$ . Then, since  $\mathbf{Q} \vdash (o' + \overline{m}) = \overline{m+1}$ ,  $(o' + y) = \overline{m+1}$ . From this we get  $\exists z (z' + y) = \overline{m+1}$ , or  $y < \overline{m+1}$ .

Hence, from each disjunct of the case for  $m$ , we can obtain the case for  $m+1$ .  $\square$

*inc:req:min:  
prop:rep-minimization*

**Proposition req.4.** *If  $\varphi_g(x, z, y)$  represents  $g(x, y)$  in  $\mathbf{Q}$ , then*

$$\varphi_f(z, y) \equiv \varphi_g(y, z, 0) \wedge \forall w (w < y \rightarrow \neg \varphi_g(w, z, 0)).$$

*represents  $f(z) = \mu x [g(x, z) = 0]$ .*

*Proof.* First we show that if  $f(n) = m$ , then  $\mathbf{Q} \vdash \varphi_f(\overline{n}, \overline{m})$ , i.e.,

$$\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, 0) \wedge \forall w (w < \overline{m} \rightarrow \neg \varphi_g(w, \overline{n}, 0)).$$

Since  $\varphi_g(x, z, y)$  represents  $g(x, z)$  and  $g(m, n) = 0$  if  $f(n) = m$ , we have

$$\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, 0).$$

If  $f(n) = m$ , then for every  $k < m$ ,  $g(k, n) \neq 0$ . So

$$\mathbf{Q} \vdash \neg \varphi_g(\overline{k}, \overline{n}, 0).$$

We get that

$$\mathbf{Q} \vdash \forall w (w < \overline{m} \rightarrow \neg \varphi_g(w, \overline{n}, 0)). \quad (6)$$

*inc:req:min:  
rep-less*

by Lemma req.2 (by (1) in case  $m = 0$  and by (2) otherwise).

Now let's show that if  $f(n) = m$ , then  $\mathbf{Q} \vdash \forall y (\varphi_f(\overline{n}, y) \rightarrow y = \overline{m})$ . We again sketch the argument informally, leaving the formalization to the reader.

Suppose  $\varphi_f(\overline{n}, y)$ . From this we get (a)  $\varphi_g(y, \overline{n}, 0)$  and (b)  $\forall w (w < y \rightarrow \neg \varphi_g(w, \overline{n}, 0))$ . By Lemma req.3,  $(y < \overline{m} \vee \overline{m} < y) \vee y = \overline{m}$ . We'll show that both  $y < \overline{m}$  and  $\overline{m} < y$  leads to a contradiction.

If  $\overline{m} < y$ , then  $\neg \varphi_g(\overline{m}, \overline{n}, 0)$  from (b). But  $m = f(n)$ , so  $g(m, n) = 0$ , and so  $\mathbf{Q} \vdash \varphi_g(\overline{m}, \overline{n}, 0)$  since  $\varphi_g$  represents  $g$ . So we have a contradiction.

Now suppose  $y < \overline{m}$ . Then since  $\mathbf{Q} \vdash \forall w (w < \overline{m} \rightarrow \neg \varphi_g(w, \overline{n}, 0))$  by eq. (6), we get  $\neg \varphi_g(y, \overline{n}, 0)$ . This again contradicts (a).  $\square$

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## Bibliography