req.1 Composition is Representable in Q

Suppose $h$ is defined by

$$h(x_0, \ldots, x_{l-1}) = f(g_0(x_0, \ldots, x_{l-1}), \ldots, g_{k-1}(x_0, \ldots, x_{l-1})).$$

where we have already found formulas $\varphi_f, \varphi_{g_0}, \ldots, \varphi_{g_{k-1}}$ representing the functions $f$, and $g_0, \ldots, g_{k-1}$, respectively. We have to find a formula $\varphi_h$ representing $h$.

Let’s start with a simple case, where all functions are 1-place, i.e., consider $h(x_0) = f(g(x_0))$. If $\varphi_f(y, z)$ represents $f$, and $\varphi_g(x, y)$ represents $g$, we need a formula $\varphi_h(x, z)$ that represents $h$. Note that $h(x) = z$ iff there is a $y$ such that both $z = f(y)$ and $y = g(x)$. (If $h(x) = z$, then $g(x)$ is such a $y$; if such a $y$ exists, then since $y = g(x)$ and $z = f(y)$, $z = f(g(x))$.) This suggests that $\exists y (\varphi_g(x, y) \land \varphi_f(y, z))$ is a good candidate for $\varphi_h(x, z)$. We just have to verify that $Q$ proves the relevant formulas.

**Proposition req.1.** If $h(n) = m$, then $Q \vdash \varphi_h(n, m)$.

*Proof.* Suppose $h(n) = m$, i.e., $f(g(n)) = m$. Let $k = g(n)$. Then

$$Q \vdash \varphi_g(n, k)$$

since $\varphi_g$ represents $g$, and

$$Q \vdash \varphi_f(k, m)$$

since $\varphi_f$ represents $f$. Thus,

$$Q \vdash \varphi_g(n, k) \land \varphi_f(k, m)$$

and consequently also

$$Q \vdash \exists y (\varphi_g(n, y) \land \varphi_f(y, m)),$$

i.e., $Q \vdash \varphi_h(n, m)$. □

**Proposition req.2.** If $h(n) = m$, then $Q \vdash \forall z (\varphi_h(n, z) \rightarrow z = m)$.

*Proof.* Suppose $h(n) = m$, i.e., $f(g(n)) = m$. Let $k = g(n)$. Then

$$Q \vdash \forall y (\varphi_g(n, y) \rightarrow y = k)$$

since $\varphi_g$ represents $g$, and

$$Q \vdash \forall z (\varphi_f(k, z) \rightarrow z = m)$$

since $\varphi_f$ represents $f$. Using just a little bit of logic, we can show that also

$$Q \vdash \forall z (\exists y (\varphi_g(n, y) \land \varphi_f(y, z)) \rightarrow z = m),$$

i.e., $Q \vdash \forall y (\varphi_h(n, y) \rightarrow y = m)$. □
The same idea works in the more complex case where \( f \) and \( g_i \) have arity greater than 1.

**Proposition req.3.** If \( \varphi_f(y_0, \ldots, y_{k-1}, z) \) represents \( f(y_0, \ldots, y_{k-1}) \) in \( Q \), and \( \varphi_{g_i}(x_0, \ldots, x_{l-1}, y) \) represents \( g_i(x_0, \ldots, x_{l-1}) \) in \( Q \), then

\[
\exists y_0, \ldots \exists y_{k-1} (\varphi_{g_0}(x_0, \ldots, x_{l-1}, y_0) \land \cdots \land \\
\varphi_{g_{k-1}}(x_0, \ldots, x_{l-1}, y_{k-1}) \land \varphi_f(y_0, \ldots, y_{k-1}, z))
\]

represents

\[
h(x_0, \ldots, x_{k-1}) = f(g_0(x_0, \ldots, x_{k-1}), \ldots, g_0(x_0, \ldots, x_{k-1})).
\]

**Proof.** Exercise.

**Problem req.1.** Using the proofs of Proposition req.2 and Proposition req.2 as a guide, carry out the proof of Proposition req.3 in detail.

Photo Credits

Bibliography