req.1 Composition is Representable in Q

inc:req:cmp:sec

Suppose h is defined by

$$h(x_0, \ldots, x_{l-1}) = f(g_0(x_0, \ldots, x_{l-1}), \ldots, g_{k-1}(x_0, \ldots, x_{l-1})).$$

where we have already found formulas $\varphi_f, \varphi_{g_0}, \ldots, \varphi_{g_{k-1}}$ representing the functions f, and g_0, …, g_{k-1}, respectively. We have to find a formula φ_h representing h.

Let's start with a simple case, where all functions are 1-place, i.e., consider $h(x) = f(g(x))$. If $\varphi_f(y, z)$ represents f, and $\varphi_g(x, y)$ represents g, we need a formula $\varphi_h(x, z)$ that represents h. Note that $h(x) = z$ iff there is a y such that both $z = f(y)$ and $y = g(x)$. (If $h(x) = z$, then $g(x)$ is such a y; if such a y exists, then since $y = g(x)$ and $z = f(y)$, $z = f(g(x))$.) This suggests that $\exists y\,(\varphi_g(x, y) \land \varphi_f(y, z))$ is a good candidate for $\varphi_h(x, z)$. We just have to verify that $\mathbf{Q}$ proves the relevant formulas.

inc:req:cmp:prop:rep1

Proposition req.1. *If $h(n) = m$, then $\mathbf{Q} \vdash \varphi_h(\overline{n}, \overline{m})$.*

Proof. Suppose $h(n) = m$, i.e., $f(g(n)) = m$. Let $k = g(n)$. Then

$$\mathbf{Q} \vdash \varphi_g(\overline{n}, \overline{k})$$

since φ_g represents g, and

$$\mathbf{Q} \vdash \varphi_f(\overline{k}, \overline{m})$$

since φ_f represents f. Thus,

$$\mathbf{Q} \vdash \varphi_g(\overline{n}, \overline{k}) \land \varphi_f(\overline{k}, \overline{m})$$

and consequently also

$$\mathbf{Q} \vdash \exists y\,(\varphi_g(\overline{n}, y) \land \varphi_f(y, \overline{m})),$$

i.e., $\mathbf{Q} \vdash \varphi_h(n, m)$. □

inc:req:cmp:prop:rep2

Proposition req.2. *If $h(n) = m$, then $\mathbf{Q} \vdash \forall z\,(\varphi_h(\overline{n}, z) \rightarrow z = \overline{m})$.*

Proof. Suppose $h(n) = m$, i.e., $f(g(n)) = m$. Let $k = g(n)$. Then

$$\mathbf{Q} \vdash \forall y\,(\varphi_g(\overline{n}, y) \rightarrow y = \overline{k})$$

since φ_g represents g, and

$$\mathbf{Q} \vdash \forall z\,(\varphi_f(\overline{k}, z) \rightarrow z = \overline{m})$$

since φ_f represents f. Using just a little bit of logic, we can show that also

$$\mathbf{Q} \vdash \forall z\,(\exists y\,(\varphi_g(\overline{n}, y) \land \varphi_f(y, z)) \rightarrow z = \overline{m}).$$

i.e., $\mathbf{Q} \vdash \forall y\,(\varphi_h(\overline{n}, y) \rightarrow y = \overline{m})$. □

The same idea works in the more complex case where f and g_i have arity greater than 1.

Proposition req.3. *If $\varphi_f(y_0, \dots, y_{k-1}, z)$ represents $f(y_0, \dots, y_{k-1})$ in $\mathbf{Q}$, and $\varphi_{g_i}(x_0, \dots, x_{l-1}, y)$ represents $g_i(x_0, \dots, x_{l-1})$ in $\mathbf{Q}$, then* inc:req:cmp: prop:rep-composition

$$\exists y_0, \dots \exists y_{k-1} \, (\varphi_{g_0}(x_0, \dots, x_{l-1}, y_0) \land \dots \land \\ \varphi_{g_{k-1}}(x_0, \dots, x_{l-1}, y_{k-1}) \land \varphi_f(y_0, \dots, y_{k-1}, z))$$

represents

$$h(x_0, \dots, x_{k-1}) = f(g_0(x_0, \dots, x_{k-1}), \dots, g_0(x_0, \dots, x_{k-1})).$$

Proof. Exercise. □

Problem req.1. Using the proofs of Proposition req.2 and Proposition req.2 as a guide, carry out the proof of Proposition req.3 in detail.

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Bibliography