

## req.1 Composition is Representable in Q

inc:req:cmp:  
sec

Suppose  $h$  is defined by

$$h(x_0, \dots, x_{l-1}) = f(g_0(x_0, \dots, x_{l-1}), \dots, g_{k-1}(x_0, \dots, x_{l-1})).$$

where we have already found formulas  $\varphi_f, \varphi_{g_0}, \dots, \varphi_{g_{k-1}}$  representing the functions  $f$ , and  $g_0, \dots, g_{k-1}$ , respectively. We have to find a formula  $\varphi_h$  representing  $h$ .

Let's start with a simple case, where all functions are 1-place, i.e., consider  $h(x) = f(g(x))$ . If  $\varphi_f(y, z)$  represents  $f$ , and  $\varphi_g(x, y)$  represents  $g$ , we need a formula  $\varphi_h(x, z)$  that represents  $h$ . Note that  $h(x) = z$  iff there is a  $y$  such that both  $z = f(y)$  and  $y = g(x)$ . (If  $h(x) = z$ , then  $g(x)$  is such a  $y$ ; if such a  $y$  exists, then since  $y = g(x)$  and  $z = f(y)$ ,  $z = f(g(x))$ .) This suggests that  $\exists y (\varphi_g(x, y) \wedge \varphi_f(y, z))$  is a good candidate for  $\varphi_h(x, z)$ . We just have to verify that Q proves the relevant formulas.

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prop:rep1

**Proposition req.1.** *If  $h(n) = m$ , then  $\mathbf{Q} \vdash \varphi_h(\bar{n}, \bar{m})$ .*

*Proof.* Suppose  $h(n) = m$ , i.e.,  $f(g(n)) = m$ . Let  $k = g(n)$ . Then

$$\mathbf{Q} \vdash \varphi_g(\bar{n}, \bar{k})$$

since  $\varphi_g$  represents  $g$ , and

$$\mathbf{Q} \vdash \varphi_f(\bar{k}, \bar{m})$$

since  $\varphi_f$  represents  $f$ . Thus,

$$\mathbf{Q} \vdash \varphi_g(\bar{n}, \bar{k}) \wedge \varphi_f(\bar{k}, \bar{m})$$

and consequently also

$$\mathbf{Q} \vdash \exists y (\varphi_g(\bar{n}, y) \wedge \varphi_f(y, \bar{m})),$$

i.e.,  $\mathbf{Q} \vdash \varphi_h(\bar{n}, \bar{m})$ . □

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prop:rep2

**Proposition req.2.** *If  $h(n) = m$ , then  $\mathbf{Q} \vdash \forall z (\varphi_h(\bar{n}, z) \rightarrow z = \bar{m})$ .*

*Proof.* Suppose  $h(n) = m$ , i.e.,  $f(g(n)) = m$ . Let  $k = g(n)$ . Then

$$\mathbf{Q} \vdash \forall y (\varphi_g(\bar{n}, y) \rightarrow y = \bar{k})$$

since  $\varphi_g$  represents  $g$ , and

$$\mathbf{Q} \vdash \forall z (\varphi_f(\bar{k}, z) \rightarrow z = \bar{m})$$

since  $\varphi_f$  represents  $f$ . Using just a little bit of logic, we can show that also

$$\mathbf{Q} \vdash \forall z (\exists y (\varphi_g(\bar{n}, y) \wedge \varphi_f(y, z)) \rightarrow z = \bar{m}).$$

i.e.,  $\mathbf{Q} \vdash \forall y (\varphi_h(\bar{n}, y) \rightarrow y = \bar{m})$ . □

The same idea works in the more complex case where  $f$  and  $g_i$  have arity greater than 1.

**Proposition req.3.** *If  $\varphi_f(y_0, \dots, y_{k-1}, z)$  represents  $f(y_0, \dots, y_{k-1})$  in  $\mathbf{Q}$ , inc:req:cmp:  
prop:rep-composition and  $\varphi_{g_i}(x_0, \dots, x_{l-1}, y)$  represents  $g_i(x_0, \dots, x_{l-1})$  in  $\mathbf{Q}$ , then*

$$\exists y_0, \dots, \exists y_{k-1} (\varphi_{g_0}(x_0, \dots, x_{l-1}, y_0) \wedge \dots \wedge \varphi_{g_{k-1}}(x_0, \dots, x_{l-1}, y_{k-1}) \wedge \varphi_f(y_0, \dots, y_{k-1}, z))$$

represents

$$h(x_0, \dots, x_{k-1}) = f(g_0(x_0, \dots, x_{k-1}), \dots, g_{k-1}(x_0, \dots, x_{k-1})).$$

*Proof.* Exercise. □

**Problem req.1.** Using the proofs of [Proposition req.2](#) and [Proposition req.2](#) as a guide, carry out the proof of [Proposition req.3](#) in detail.

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## Bibliography