

req.1 Basic Functions are Representable in \mathbf{Q}

inc:req:bre: sec First we have to show that all the basic functions are representable in \mathbf{Q} . In the end, we need to show how to assign to each k -ary basic function $f(x_0, \dots, x_{k-1})$ a formula $\varphi_f(x_0, \dots, x_{k-1}, y)$ that represents it.

We will be able to represent zero, successor, plus, times, the characteristic function for equality, and projections. In each case, the appropriate representing function is entirely straightforward; for example, zero is represented by the formula $y = 0$, successor is represented by the formula $x'_0 = y$, and addition is represented by the formula $(x_0 + x_1) = y$. The work involves showing that \mathbf{Q} can prove the relevant sentences; for example, saying that addition is represented by the formula above involves showing that for every pair of natural numbers m and n , \mathbf{Q} proves

$$\begin{aligned} \bar{n} + \bar{m} &= \overline{n + m} \text{ and} \\ \forall y ((\bar{n} + \bar{m}) = y &\rightarrow y = \overline{n + m}). \end{aligned}$$

inc:req:bre: prop:rep-zero **Proposition req.1.** *The zero function $\text{zero}(x) = 0$ is represented in \mathbf{Q} by $\varphi_{\text{zero}}(x, y) \equiv y = 0$.*

inc:req:bre: prop:rep-succ **Proposition req.2.** *The successor function $\text{succ}(x) = x + 1$ is represented in \mathbf{Q} by $\varphi_{\text{succ}}(x, y) \equiv y = x'$.*

inc:req:bre: prop:rep-proj **Proposition req.3.** *The projection function $P_i^n(x_0, \dots, x_{n-1}) = x_i$ is represented in \mathbf{Q} by*

$$\varphi_{P_i^n}(x_0, \dots, x_{n-1}, y) \equiv y = x_i.$$

Problem req.1. Prove that $y = 0$, $y = x'$, and $y = x_i$ represent zero, succ, and P_i^n , respectively.

inc:req:bre: prop:rep-id **Proposition req.4.** *The characteristic function of $=$,*

$$\chi_{=} (x_0, x_1) = \begin{cases} 1 & \text{if } x_0 = x_1 \\ 0 & \text{otherwise} \end{cases}$$

is represented in \mathbf{Q} by

$$\varphi_{\chi_{=}}(x_0, x_1, y) \equiv (x_0 = x_1 \wedge y = \bar{1}) \vee (x_0 \neq x_1 \wedge y = \bar{0}).$$

The proof requires the following lemma.

inc:req:bre: lem:q-proves-neq **Lemma req.5.** *Given natural numbers n and m , if $n \neq m$, then $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$.*

Proof. Use induction on n to show that for every m , if $n \neq m$, then $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$.

In the base case, $n = 0$. If m is not equal to 0, then $m = k + 1$ for some natural number k . We have an axiom that says $\forall x 0 \neq x'$. By a quantifier axiom, replacing x by \bar{k} , we can conclude $0 \neq \bar{k}'$. But \bar{k}' is just \bar{m} .

In the induction step, we can assume the claim is true for n , and consider $n + 1$. Let m be any natural number. There are two possibilities: either $m = 0$ or for some k we have $m = k + 1$. The first case is handled as above. In the second case, suppose $n + 1 \neq k + 1$. Then $n \neq k$. By the induction hypothesis for n we have $\mathbf{Q} \vdash \bar{n} \neq \bar{k}$. We have an axiom that says $\forall x \forall y x' = y' \rightarrow x = y$. Using a quantifier axiom, we have $\bar{n}' = \bar{k}' \rightarrow \bar{n} = \bar{k}$. Using propositional logic, we can conclude, in \mathbf{Q} , $\bar{n} \neq \bar{k} \rightarrow \bar{n}' \neq \bar{k}'$. Using modus ponens, we can conclude $\bar{n}' \neq \bar{k}'$, which is what we want, since \bar{k}' is \bar{m} . \square

explanation

Note that the lemma does not say much: in essence it says that \mathbf{Q} can prove that different numerals denote different objects. For example, \mathbf{Q} proves $0'' \neq 0'''$. But showing that this holds in general requires some care. Note also that although we are using induction, it is induction *outside* of \mathbf{Q} .

Proof of Proposition req.4. If $n = m$, then \bar{n} and \bar{m} are the same term, and $\chi_{=}(n, m) = 1$. But $\mathbf{Q} \vdash (\bar{n} = \bar{m} \wedge \bar{1} = \bar{1})$, so it proves $\varphi_{=}(n, m, \bar{1})$. If $n \neq m$, then $\chi_{=}(n, m) = 0$. By Lemma req.5, $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$ and so also $(\bar{n} \neq \bar{m} \wedge 0 = 0)$. Thus $\mathbf{Q} \vdash \varphi_{=}(n, m, \bar{0})$.

For the second part, we also have two cases. If $n = m$, we have to show that $\mathbf{Q} \vdash \forall y (\varphi_{=}(n, m, y) \rightarrow y = \bar{1})$. Arguing informally, suppose $\varphi_{=}(n, m, y)$, i.e.,

$$(\bar{n} = \bar{n} \wedge y = \bar{1}) \vee (\bar{n} \neq \bar{n} \wedge y = \bar{0})$$

The left disjunct implies $y = \bar{1}$ by logic; the right contradicts $\bar{n} = \bar{n}$ which is provable by logic.

Suppose, on the other hand, that $n \neq m$. Then $\varphi_{=}(n, m, y)$ is

$$(\bar{n} = \bar{m} \wedge y = \bar{1}) \vee (\bar{n} \neq \bar{m} \wedge y = \bar{0})$$

Here, the left disjunct contradicts $\bar{n} \neq \bar{m}$, which is provable in \mathbf{Q} by Lemma req.5; the right disjunct entails $y = \bar{0}$. \square

Proposition req.6. *The addition function $\text{add}(x_0, x_1) = x_0 + x_1$ is represented in \mathbf{Q} by*

$$\varphi_{\text{add}}(x_0, x_1, y) \equiv y = (x_0 + x_1).$$

inc:req:bre:
prop:rep-add

Lemma req.7. $\mathbf{Q} \vdash (\bar{n} + \bar{m}) = \overline{n + m}$

inc:req:bre:
lem:q-proves-add

Proof. We prove this by induction on m . If $m = 0$, the claim is that $\mathbf{Q} \vdash (\bar{n} + 0) = \bar{n}$. This follows by axiom Q_4 . Now suppose the claim for m ; let's prove the claim for $m + 1$, i.e., prove that $\mathbf{Q} \vdash (\bar{n} + \overline{m + 1}) = \overline{n + m + 1}$. Note that $\overline{m + 1}$ is just \bar{m}' , and $\overline{n + m + 1}$ is just $\overline{n + m}'$. By axiom Q_5 , $\mathbf{Q} \vdash (\bar{n} + \bar{m}') = \overline{(n + m)'}$. By induction hypothesis, $\mathbf{Q} \vdash (\bar{n} + \bar{m}) = \overline{n + m}$. So $\mathbf{Q} \vdash (\bar{n} + \bar{m}') = \overline{n + m}'$. \square

Proof of Proposition req.6. The formula $\varphi_{\text{add}}(x_0, x_1, y)$ representing add is $y = (x_0 + x_1)$. First we show that if $\text{add}(n, m) = k$, then $\mathbf{Q} \vdash \varphi_{\text{add}}(\bar{n}, \bar{m}, \bar{k})$, i.e.,

$\mathbf{Q} \vdash \bar{k} = (\bar{n} + \bar{m})$. But since $k = n + m$, \bar{k} just is $\overline{n + m}$, and we've shown in [Lemma req.7](#) that $\mathbf{Q} \vdash (\bar{n} + \bar{m}) = \overline{n + m}$.

We also have to show that if $\text{add}(n, m) = k$, then

$$\mathbf{Q} \vdash \forall y (\varphi_{\text{add}}(\bar{n}, \bar{m}, y) \rightarrow y = \bar{k}).$$

Suppose we have $(\bar{n} + \bar{m}) = y$. Since

$$\mathbf{Q} \vdash (\bar{n} + \bar{m}) = \overline{n + m},$$

we can replace the left side with $\overline{n + m}$ and get $\overline{n + m} = y$, for arbitrary y . \square

[inc:req:bre:](#)
[prop:rep-mult](#)

Proposition req.8. *The multiplication function $\text{mult}(x_0, x_1) = x_0 \cdot x_1$ is represented in \mathbf{Q} by*

$$\varphi_{\text{mult}}(x_0, x_1, y) \equiv y = (x_0 \times x_1).$$

Proof. Exercise. \square

[inc:req:bre:](#)
[lem:q-proves-mult](#)

Lemma req.9. $\mathbf{Q} \vdash (\bar{n} \times \bar{m}) = \overline{n \cdot m}$

Proof. Exercise. \square

Problem req.2. Prove [Lemma req.9](#).

Problem req.3. Use [Lemma req.9](#) to prove [Proposition req.8](#).

Recall that we use \times for the function symbol of the language of arithmetic, [explanation](#) and \cdot for the ordinary multiplication operation on numbers. So \cdot can appear between expressions for numbers (such as in $m \cdot n$) while \times appears only between terms of the language of arithmetic (such as in $(\bar{m} \times \bar{n})$). Even more confusingly, $+$ is used for both the [function symbol](#) and the addition operation. When it appears between terms—e.g., in $(\bar{n} + \bar{m})$ —it is the 2-place [function symbol](#) of the language of arithmetic, and when it appears between numbers—e.g., in $n + m$ —it is the addition operation. This includes the case $\overline{n + m}$: this is the standard numeral corresponding to the number $n + m$.

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Bibliography