Basic Functions are Representable in Q

First we have to show that all the basic functions are representable in Q. In the end, we need to show how to assign to each k-ary basic function \( f(x_0, \ldots, x_{k-1}) \) a formula \( \varphi_f(x_0, \ldots, x_{k-1}, y) \) that represents it.

We will be able to represent zero, successor, plus, times, the characteristic function for equality, and projections. In each case, the appropriate representing function is entirely straightforward; for example, zero is represented by the formula \( y = 0 \), successor is represented by the formula \( x' = y \), and addition is represented by the formula \( (x_0 + x_1) = y \). The work involves showing that Q can prove the relevant sentences; for example, saying that addition is represented by the formula above involves showing that for every pair of natural numbers \( m \) and \( n \), Q proves

\[
\pi + m = \overline{n + m} \quad \text{and} \quad \forall y \ ((\pi + m) = y \rightarrow y = \overline{n + m}).
\]

**Proposition req.1.** The zero function \( \text{zero}(x) = 0 \) is represented in Q by \( y = 0 \).

**Proposition req.2.** The successor function \( \text{succ}(x) = x + 1 \) is represented in Q by \( y = x' \).

**Proposition req.3.** The projection function \( P^n_i(x_0, \ldots, x_{n-1}) = x_i \) is represented in Q by \( y = x_i \).

**Problem req.1.** Prove that \( y = 0 \), \( y = x' \), and \( y = x_i \) represent zero, succ, and \( P^n_i \), respectively.

**Proposition req.4.** The characteristic function of \( = \),

\[
\chi_=(x_0, x_1) = \begin{cases} 
1 & \text{if } x_0 = x_1 \\
0 & \text{otherwise}
\end{cases}
\]

is represented in Q by

\[
(x_0 = x_1 \land y = \overline{1}) \lor (x_0 \neq x_1 \land y = \overline{0}).
\]

The proof requires the following lemma.

**Lemma req.5.** Given natural numbers \( n \) and \( m \), if \( n \neq m \), then \( Q \vdash \pi \neq \overline{m} \).

**Proof.** Use induction on \( n \) to show that for every \( m \), if \( n \neq m \), then \( Q \vdash \pi \neq \overline{m} \).

In the base case, \( n = 0 \). If \( m \) is not equal to 0, then \( m = k + 1 \) for some natural number \( k \). We have an axiom that says \( \forall x \ 0 \neq x' \). By a quantifier axiom, replacing \( x \) by \( \overline{k} \), we can conclude \( 0 \neq \overline{k} \). But \( \overline{k} \) is just \( \overline{m} \).

In the induction step, we can assume the claim is true for \( n \), and consider \( n + 1 \). Let \( m \) be any natural number. There are two possibilities: either \( m = 0 \)
or for some k we have \( m = k + 1 \). The first case is handled as above. In the second case, suppose \( n + 1 \neq k + 1 \). Then \( n \neq k \). By the induction hypothesis for \( n \) we have \( Q \vdash \pi \neq k \). We have an axiom that says \( \forall x \forall y. x' = y' \rightarrow x = y \). Using a quantifier axiom, we have \( \pi = k \rightarrow \pi = k \). Using propositional logic, we can conclude, in \( Q \), \( \pi \neq k \rightarrow \pi \neq k' \). Using modus ponens, we can conclude \( \pi' \neq k' \), which is what we want, since \( k' \) is \( m \).

Note that the lemma does not say much: in essence it says that \( Q \) can prove that different numerals denote different objects. For example, \( Q \) proves \( 0'' \neq 0''' \). But showing that this holds in general requires some care. Note also that although we are using induction, it is induction outside of \( Q \).

**Proof of Proposition req.4.** If \( n = m \), then \( \overline{n} \) and \( \overline{m} \) are the same term, and \( \chi_\equiv(n, m) = 1 \). But \( Q \vdash (\pi = \overline{m} \land \overline{1} = \overline{1}) \), so it proves \( \varphi_\equiv(\pi, \overline{m}, \overline{1}) \). If \( n \neq m \), then \( \chi_\equiv(n, m) = 0 \). By Lemma req.5, \( Q \vdash \pi \neq \overline{m} \) and so also (\( \pi \neq m \land o = 0 \)).

Thus \( Q \vdash \varphi_\equiv(\pi, \overline{m}, \overline{0}) \).

For the second part, we also have two cases. If \( n = m \), we have to show that \( Q \vdash \forall \varphi_\equiv(\pi, \overline{m}, y) \rightarrow y = \overline{1} \). Arguing informally, suppose \( \varphi_\equiv(\pi, \overline{m}, y) \), i.e.,

\[ (\pi = \overline{n} \land y = \overline{1}) \lor (\pi \neq \overline{m} \land y = \overline{0}) \]

The left disjunct implies \( y = \overline{1} \) by logic; the right contradicts \( \pi = \overline{n} \) which is provable by logic.

Suppose, on the other hand, that \( n \neq m \). Then \( \varphi_\equiv(\pi, \overline{m}, y) \) is

\[ (\pi = \overline{m} \land y = \overline{1}) \lor (\pi \neq \overline{m} \land y = \overline{0}) \]

Here, the left disjunct contradicts \( \pi \neq \overline{m} \), which is provable in \( Q \) by Lemma req.5; the right disjunct entails \( y = \overline{0} \).

**Proposition req.6.** The addition function \( \text{add}(x_0, x_1) = x_0 + x_1 \) is represented in \( Q \) by

\[ y = (x_0 + x_1). \]

**Lemma req.7.** \( Q \vdash (\pi + m) = \overline{n + m} \)

**Proof.** We prove this by induction on \( m \). If \( m = 0 \), the claim is that \( Q \vdash (\pi + 0) = \pi \). This follows by axiom Q4. Now suppose the claim for \( m \); let’s prove the claim for \( m + 1 \), i.e., prove that \( Q \vdash (\pi + (m + 1)) = \overline{n + m + 1} \). Note that \( m + 1 \) is just \( m' \), and \( n + m + 1 \) is just \( n + m \). By axiom Q5, \( Q \vdash (\pi + m') = (\pi + m') \). By induction hypothesis, \( Q \vdash (\pi + m) = \overline{n + m} \). So \( Q \vdash (\pi + m') = \overline{n + m} \).

**Proof of Proposition req.6.** The formula \( \varphi_{\text{add}}(x_0, x_1, y) \) representing \( \text{add} \) is \( (x_0 + x_1) \). First we show that if \( \text{add}(n, m) = k \), then \( Q \vdash \varphi_{\text{add}}(\pi, \overline{m}, \overline{k}) \), i.e., \( Q \vdash \pi = (\pi + m) \). But since \( k = n + m \), \( k \) just is \( n + m \), and we’ve shown in Lemma req.7 that \( Q \vdash (\pi + m) = \overline{n + m} \).
We also have to show that if $\text{add}(n, m) = k$, then

$$Q \models \forall y (\varphi_{\text{add}}(\overline{n}, \overline{m}, y) \rightarrow y = \overline{k}).$$

Suppose we have $(\overline{n} + \overline{m}) = \overline{y}$. Since

$$Q \models (\overline{n} + \overline{m}) = \overline{n + m},$$

we can replace the left side with $\overline{n + m}$ and get $\overline{n + m} = \overline{y}$, for arbitrary $\overline{y}$. □

**Proposition req.8.** The multiplication function $\text{mult}(x_0, x_1) = x_0 \cdot x_1$ is represented in $Q$ by

$$y = (x_0 \times x_1).$$

*Proof.* Exercise. □

**Lemma req.9.** $Q \vdash (\overline{n} \times \overline{m}) = \overline{n \cdot m}$

*Proof.* Exercise. □

**Problem req.2.** Prove Lemma req.9.

**Problem req.3.** Use Lemma req.9 to prove Proposition req.8.

Recall that we use $\times$ for the function symbol of the language of arithmetic, and $\cdot$ for the ordinary multiplication operation on numbers. So $\cdot$ can appear between expressions for numbers (such as in $m \cdot n$) while $\times$ appears only between terms of the language of arithmetic (such as in $(\overline{m} \times \overline{n})$). Even more confusingly, $+$ is used for both the function symbol and the addition operation. When it appears between terms—e.g., in $(\overline{n} + \overline{m})$—it is the 2-place function symbol of the language of arithmetic, and when it appears between numbers—e.g., in $n + m$—it is the addition operation. This includes the case $\overline{n + m}$: this is the standard numeral corresponding to the number $n + m$.

**Photo Credits**

**Bibliography**