

## req.1 Basic Functions are Representable in $\mathbf{Q}$

inc:req:bre:sec First we have to show that all the basic functions are representable in  $\mathbf{Q}$ . In the end, we need to show how to assign to each  $k$ -ary basic function  $f(x_0, \dots, x_{k-1})$  a formula  $\varphi_f(x_0, \dots, x_{k-1}, y)$  that represents it.

We will be able to represent zero, successor, plus, times, the characteristic function for equality, and projections. In each case, the appropriate representing function is entirely straightforward; for example, zero is represented by the formula  $y = 0$ , successor is represented by the formula  $x'_0 = y$ , and addition is represented by the formula  $(x_0 + x_1) = y$ . The work involves showing that  $\mathbf{Q}$  can prove the relevant sentences; for example, saying that addition is represented by the formula above involves showing that for every pair of natural numbers  $m$  and  $n$ ,  $\mathbf{Q}$  proves

$$\begin{aligned} \bar{n} + \bar{m} &= \overline{n + m} \text{ and} \\ \forall y ((\bar{n} + \bar{m}) = y &\rightarrow y = \overline{n + m}). \end{aligned}$$

inc:req:bre:prop:rep-zero **Proposition req.1.** *The zero function  $\text{zero}(x) = 0$  is represented in  $\mathbf{Q}$  by  $y = 0$ .*

inc:req:bre:prop:rep-succ **Proposition req.2.** *The successor function  $\text{succ}(x) = x + 1$  is represented in  $\mathbf{Q}$  by  $y = x'$ .*

inc:req:bre:prop:rep-proj **Proposition req.3.** *The projection function  $P_i^n(x_0, \dots, x_{n-1}) = x_i$  is represented in  $\mathbf{Q}$  by  $y = x_i$ .*

**Problem req.1.** Prove that  $y = 0$ ,  $y = x'$ , and  $y = x_i$  represent zero, succ, and  $P_i^n$ , respectively.

inc:req:bre:prop:rep-id **Proposition req.4.** *The characteristic function of  $=$ ,*

$$\chi_{=} (x_0, x_1) = \begin{cases} 1 & \text{if } x_0 = x_1 \\ 0 & \text{otherwise} \end{cases}$$

*is represented in  $\mathbf{Q}$  by*

$$(x_0 = x_1 \wedge y = \bar{1}) \vee (x_0 \neq x_1 \wedge y = \bar{0}).$$

The proof requires the following lemma.

inc:req:bre:lem:q-proves-neq **Lemma req.5.** *Given natural numbers  $n$  and  $m$ , if  $n \neq m$ , then  $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$ .*

*Proof.* Use induction on  $n$  to show that for every  $m$ , if  $n \neq m$ , then  $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$ .

In the base case,  $n = 0$ . If  $m$  is not equal to 0, then  $m = k + 1$  for some natural number  $k$ . We have an axiom that says  $\forall x 0 \neq x'$ . By a quantifier axiom, replacing  $x$  by  $\bar{k}$ , we can conclude  $0 \neq \bar{k}'$ . But  $\bar{k}'$  is just  $\bar{m}$ .

In the induction step, we can assume the claim is true for  $n$ , and consider  $n + 1$ . Let  $m$  be any natural number. There are two possibilities: either  $m = 0$

or for some  $k$  we have  $m = k + 1$ . The first case is handled as above. In the second case, suppose  $n + 1 \neq k + 1$ . Then  $n \neq k$ . By the induction hypothesis for  $n$  we have  $\mathbf{Q} \vdash \bar{n} \neq \bar{k}$ . We have an axiom that says  $\forall x \forall y x' = y' \rightarrow x = y$ . Using a quantifier axiom, we have  $\bar{n}' = \bar{k}' \rightarrow \bar{n} = \bar{k}$ . Using propositional logic, we can conclude, in  $\mathbf{Q}$ ,  $\bar{n} \neq \bar{k} \rightarrow \bar{n}' \neq \bar{k}'$ . Using modus ponens, we can conclude  $\bar{n}' \neq \bar{k}'$ , which is what we want, since  $\bar{k}'$  is  $\bar{m}$ .  $\square$

explanation

Note that the lemma does not say much: in essence it says that  $\mathbf{Q}$  can prove that different numerals denote different objects. For example,  $\mathbf{Q}$  proves  $0'' \neq 0'''$ . But showing that this holds in general requires some care. Note also that although we are using induction, it is induction *outside* of  $\mathbf{Q}$ .

*Proof of Proposition req.4.* If  $n = m$ , then  $\bar{n}$  and  $\bar{m}$  are the same term, and  $\chi_{=} (n, m) = 1$ . But  $\mathbf{Q} \vdash (\bar{n} = \bar{m} \wedge \bar{1} = \bar{1})$ , so it proves  $\varphi_{=} (\bar{n}, \bar{m}, \bar{1})$ . If  $n \neq m$ , then  $\chi_{=} (n, m) = 0$ . By Lemma req.5,  $\mathbf{Q} \vdash \bar{n} \neq \bar{m}$  and so also  $(\bar{n} \neq \bar{m} \wedge 0 = 0)$ . Thus  $\mathbf{Q} \vdash \varphi_{=} (\bar{n}, \bar{m}, \bar{0})$ .

For the second part, we also have two cases. If  $n = m$ , we have to show that that  $\mathbf{Q} \vdash \forall (\varphi_{=} (\bar{n}, \bar{m}, y) \rightarrow y = \bar{1})$ . Arguing informally, suppose  $\varphi_{=} (\bar{n}, \bar{m}, y)$ , i.e.,

$$(\bar{n} = \bar{n} \wedge y = \bar{1}) \vee (\bar{n} \neq \bar{n} \wedge y = \bar{0})$$

The left disjunct implies  $y = \bar{1}$  by logic; the right contradicts  $\bar{n} = \bar{n}$  which is provable by logic.

Suppose, on the other hand, that  $n \neq m$ . Then  $\varphi_{=} (\bar{n}, \bar{m}, y)$  is

$$(\bar{n} = \bar{m} \wedge y = \bar{1}) \vee (\bar{n} \neq \bar{m} \wedge y = \bar{0})$$

Here, the left disjunct contradicts  $\bar{n} \neq \bar{m}$ , which is provable in  $\mathbf{Q}$  by Lemma req.5; the right disjunct entails  $y = \bar{0}$ .  $\square$

**Proposition req.6.** *The addition function  $\text{add}(x_0, x_1) = x_0 + x_1$  is represented in  $\mathbf{Q}$  by*

$$y = (x_0 + x_1).$$

inc:req:bre:  
prop:rep-add

**Lemma req.7.**  $\mathbf{Q} \vdash (\bar{n} + \bar{m}) = \overline{n + m}$

inc:req:bre:  
lem:q-proves-add

*Proof.* We prove this by induction on  $m$ . If  $m = 0$ , the claim is that  $\mathbf{Q} \vdash (\bar{n} + 0) = \bar{n}$ . This follows by axiom  $Q_4$ . Now suppose the claim for  $m$ ; let's prove the claim for  $m + 1$ , i.e., prove that  $\mathbf{Q} \vdash (\bar{n} + \bar{m} + \bar{1}) = \overline{n + m + 1}$ . Note that  $\bar{m} + \bar{1}$  is just  $\bar{m}'$ , and  $\overline{n + m + 1}$  is just  $\overline{n + m'}$ . By axiom  $Q_5$ ,  $\mathbf{Q} \vdash (\bar{n} + \bar{m}') = \overline{(n + m')}$ . By induction hypothesis,  $\mathbf{Q} \vdash (\bar{n} + \bar{m}) = \overline{n + m}$ . So  $\mathbf{Q} \vdash (\bar{n} + \bar{m}') = \overline{n + m'}$ .  $\square$

*Proof of Proposition req.6.* The formula  $\varphi_{\text{add}}(x_0, x_1, y)$  representing  $\text{add}$  is  $y = (x_0 + x_1)$ . First we show that if  $\text{add}(n, m) = k$ , then  $\mathbf{Q} \vdash \varphi_{\text{add}}(\bar{n}, \bar{m}, \bar{k})$ , i.e.,  $\mathbf{Q} \vdash \bar{k} = (\bar{n} + \bar{m})$ . But since  $k = n + m$ ,  $\bar{k}$  just is  $\overline{n + m}$ , and we've shown in Lemma req.7 that  $\mathbf{Q} \vdash (\bar{n} + \bar{m}) = \overline{n + m}$ .

We also have to show that if  $\text{add}(n, m) = k$ , then

$$\mathbf{Q} \vdash \forall y (\varphi_{\text{add}}(\bar{n}, \bar{m}, y) \rightarrow y = \bar{k}).$$

Suppose we have  $(\bar{n} + \bar{m}) = y$ . Since

$$\mathbf{Q} \vdash (\bar{n} + \bar{m}) = \overline{n + m},$$

we can replace the left side with  $\overline{n + m}$  and get  $\overline{n + m} = y$ , for arbitrary  $y$ .  $\square$

*inc:req:bre: prop:rep-mult* **Proposition req.8.** *The multiplication function  $\text{mult}(x_0, x_1) = x_0 \cdot x_1$  is represented in  $\mathbf{Q}$  by*

$$y = (x_0 \times x_1).$$

*Proof.* Exercise.  $\square$

*inc:req:bre: lem:q-proves-mult* **Lemma req.9.**  $\mathbf{Q} \vdash (\bar{n} \times \bar{m}) = \overline{n \cdot m}$

*Proof.* Exercise.  $\square$

**Problem req.2.** Prove [Lemma req.9](#).

**Problem req.3.** Use [Lemma req.9](#) to prove [Proposition req.8](#).

Recall that we use  $\times$  for the function symbol of the language of arithmetic, [explanation](#) and  $\cdot$  for the ordinary multiplication operation on numbers. So  $\cdot$  can appear between expressions for numbers (such as in  $m \cdot n$ ) while  $\times$  appears only between terms of the language of arithmetic (such as in  $(\bar{m} \times \bar{n})$ ). Even more confusingly,  $+$  is used for both the [function symbol](#) and the addition operation. When it appears between terms—e.g., in  $(\bar{n} + \bar{m})$ —it is the 2-place [function symbol](#) of the language of arithmetic, and when it appears between numbers—e.g., in  $n + m$ —it is the operation. This includes the case  $\overline{n + m}$ : this is the standard numeral corresponding to the number  $n + m$ .

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## Bibliography