Basic Functions are Representable in \( Q \)

First we have to show that all the basic functions are representable in \( Q \). In the end, we need to show how to assign to each \( k \)-ary basic function \( f(x_0, \ldots, x_{k-1}) \) a formula \( \varphi_f(x_0, \ldots, x_{k-1}, y) \) that represents it.

We will be able to represent zero, successor, plus, times, the characteristic function for equality, and projections. In each case, the appropriate representing function is entirely straightforward; for example, zero is represented by the formula \( y = 0 \), successor is represented by the formula \( x_0' = y \), and addition is represented by the formula \( (x_0 + x_1) = y \). The work involves showing that \( Q \) can prove the relevant sentences; for example, saying that addition is represented by the formula above involves showing that for every pair of natural numbers \( m \) and \( n \), \( Q \) proves

\[
\forall y ((\overline{n + m}) = y \Rightarrow y = \overline{n + m}).
\]

**Proposition req.1.** The zero function \( \text{zero}(x) = 0 \) is represented in \( Q \) by \( \varphi_{\text{zero}}(x, y) \equiv y = 0 \).

**Proposition req.2.** The successor function \( \text{succ}(x) = x + 1 \) is represented in \( Q \) by \( \varphi_{\text{succ}}(x, y) \equiv y = x' \).

**Proposition req.3.** The projection function \( P_i^n(x_0, \ldots, x_{n-1}) = x_i \) is represented in \( Q \) by \( \varphi_{P_i^n}(x_0, \ldots, x_{n-1}, y) \equiv y = x_i \).

**Problem req.1.** Prove that \( y = 0 \), \( y = x' \), and \( y = x_i \) represent zero, succ, and \( P_i^n \), respectively.

**Proposition req.4.** The characteristic function of \( = \),

\[
\chi_=(x_0, x_1) = \begin{cases} 1 & \text{if } x_0 = x_1 \\ 0 & \text{otherwise} \end{cases}
\]

is represented in \( Q \) by

\[
\varphi_{\chi_=}(x_0, x_1, y) \equiv (x_0 = x_1 \land y = \top) \lor (x_0 \neq x_1 \land y = \bot).
\]

The proof requires the following lemma.

**Lemma req.5.** Given natural numbers \( n \) and \( m \), if \( n \neq m \), then \( Q \vdash \overline{n} \neq \overline{m} \).

**Proof.** Use induction on \( n \) to show that for every \( m \), if \( n \neq m \), then \( Q \vdash \overline{n} \neq \overline{m} \).

In the base case, \( n = 0 \). If \( m \) is not equal to 0, then \( m = k + 1 \) for some natural number \( k \). We have an axiom that says \( \forall x \neq x' \). By a quantifier axiom, replacing \( x \) by \( \overline{k} \), we can conclude \( 0 \neq \overline{k} \). But \( \overline{k} \) is just \( \overline{m} \).

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In the induction step, we can assume the claim is true for \( n \), and consider \( n + 1 \). Let \( m \) be any natural number. There are two possibilities: either \( m = 0 \) or for some \( k \) we have \( m = k + 1 \). The first case is handled as above. In the second case, suppose \( n + 1 \neq k + 1 \). Then \( n \neq k \). By the induction hypothesis for \( n \) we have \( Q \vdash \pi \neq \overline{k} \). We have an axiom that says \( \forall x \forall y \, x' = y' \rightarrow x = y \).

Using a quantifier axiom, we have \( Q \vdash \pi \neq \overline{k} \rightarrow \pi' \neq \overline{k}' \). Using propositional logic, we can conclude, in \( Q \), \( \pi \neq \overline{k} \rightarrow \pi' \neq \overline{k}' \). Using modus ponens, we can conclude \( \pi' \neq \overline{k}' \), which is what we want, since \( \overline{k}' \) is \( \overline{m} \).

Note that the lemma does not say much: in essence it says that \( Q \) can prove that different numerals denote different objects. For example, \( Q \) proves \( 0'' \neq 0''' \). But showing that this holds in general requires some care. Note also that although we are using induction, it is induction outside of \( Q \).

**Proof of Proposition req.4.** If \( n = m \), then \( \pi \) and \( \overline{m} \) are the same term, and \( \chi = (n, m) = 1 \). But \( Q \vdash (\pi = \overline{m} \land \top = \top) \), so it proves \( \varphi = (\pi, \overline{m}, \top) \). If \( n \neq m \), then \( \chi = (n, m) = 0 \). By Lemma req.5, \( Q \vdash \pi \neq \overline{m} \) and so also \( (\pi \neq \overline{m} \land 0 = 0) \). Thus \( Q \vdash \varphi = (\pi, \overline{m}, 0) \).

For the second part, we also have two cases. If \( n = m \), we have to show that \( Q \vdash \forall y \, (\varphi = (\pi, \overline{m}, y) \rightarrow y = \top) \). Arguing informally, suppose \( \varphi = (\pi, \overline{m}, y) \), i.e.,

\[
(\pi = \pi \land y = \top) \lor (\pi \neq \pi \land y = \top)
\]

The left disjunct implies \( y = \top \) by logic; the right contradicts \( \pi = \pi \) which is provable by logic.

Suppose, on the other hand, that \( n \neq m \). Then \( \varphi = (\pi, \overline{m}, y) \) is

\[
(\pi = \pi \land y = \top) \lor (\pi \neq \pi \land y = \top)
\]

Here, the left disjunct contradicts \( \pi \neq \overline{m} \), which is provable in \( Q \) by Lemma req.5; the right disjunct entails \( y = \top \).

**Proposition req.6.** The addition function \( \text{add}(x_0, x_1) = x_0 + x_1 \) is represented in \( Q \) by

\[
\varphi_{\text{add}}(x_0, x_1, y) \equiv y = (x_0 + x_1).
\]

**Lemma req.7.** \( Q \vdash (\pi + \overline{m}) = \overline{n + m} \)

**Proof.** We prove this by induction on \( m \). If \( m = 0 \), the claim is that \( Q \vdash (\pi + \overline{0}) = \pi \). This follows by axiom \( Q_4 \). Now suppose the claim for \( m \), i.e., prove that \( Q \vdash (\pi + \overline{m + 1}) = \overline{n + m + 1} \). Note that \( \overline{m + 1} \) is just \( \overline{m} ' \), and \( \overline{n + m + 1} \) is just \( \overline{n + m} \). By axiom \( Q_5 \), \( Q \vdash (\pi + \overline{m'}) = (\overline{n + m}) ' \). By induction hypothesis, \( Q \vdash (\pi + \overline{m}) = \overline{n + m} \). So \( Q \vdash (\pi + \overline{m}) = \overline{n + m} \). □

**Proof of Proposition req.6.** The formula \( \varphi_{\text{add}}(x_0, x_1, y) \) representing add is \( y = (x_0 + x_1) \). First we show that if \( \text{add}(n, m) = k \), then \( Q \vdash \varphi_{\text{add}}(\pi, \overline{m}, \overline{k}) \), i.e.,

\[
\varphi = (\pi, \overline{m}, \overline{k}) \]

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\[ Q \vdash \bar{k} = (\bar{n} + \bar{m}). \] But since \( k = n + m \), \( \bar{k} \) just is \( \bar{n} + \bar{m} \), and we’ve shown in Lemma req.7 that \( Q \vdash (\bar{n} + \bar{m}) = \bar{n} + \bar{m} \).

We also have to show that if \( \text{add}(n, m) = k \), then
\[ Q \vdash \forall y \ (\varphi_{\text{add}}(\bar{n}, \bar{m}, y) \rightarrow y = \bar{k}). \]

Suppose we have \( (\bar{n} + \bar{m}) = y \). Since
\[ Q \vdash (\bar{n} + \bar{m}) = \bar{n} + \bar{m}, \]
we can replace the left side with \( \bar{n} + \bar{m} \) and get \( \bar{n} + \bar{m} = y \), for arbitrary \( y \). \( \square \)

**Proposition req.8.** The multiplication function \( \text{mult}(x_0, x_1) = x_0 \cdot x_1 \) is represented in \( Q \) by
\[ \varphi_{\text{mult}}(x_0, x_1, y) \equiv y = (x_0 \times x_1). \]

**Proof.** Exercise. \( \square \)

**Lemma req.9.** \( Q \vdash (\bar{n} \times \bar{m}) = \bar{n} \cdot \bar{m} \)

**Proof.** Exercise. \( \square \)

**Problem req.2.** Prove Lemma req.9.

**Problem req.3.** Use Lemma req.9 to prove Proposition req.8.

Recall that we use \( \times \) for the function symbol of the language of arithmetic, and \( \cdot \) for the ordinary multiplication operation on numbers. So \( \cdot \) can appear between expressions for numbers (such as in \( m \cdot n \)) while \( \times \) appears only between terms of the language of arithmetic (such as in \( (\bar{n} \times \bar{m}) \)). Even more confusingly, \( + \) is used for both the function symbol and the addition operation. When it appears between terms—e.g., in \( (\bar{n} + \bar{m}) \)—it is the 2-place function symbol of the language of arithmetic, and when it appears between numbers—e.g., in \( n + m \)—it is the addition operation. This includes the case \( \bar{n} + \bar{m} \): this is the standard numeral corresponding to the number \( n + m \).

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**Bibliography**