

## int.1 Undecidability and Incompleteness

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sec

Gödel's proof of the incompleteness theorems require arithmetization of syntax. But even without that we can obtain some nice results just on the assumption that a theory **represents** all **decidable** relations. The proof is a diagonal argument similar to the proof of the undecidability of the halting problem.

**Theorem int.1.** *If  $\Gamma$  is a consistent theory that **represents** every **decidable** relation, then  $\Gamma$  is not **decidable**.*

*Proof.* Suppose  $\Gamma$  were **decidable**. We show that if  $\Gamma$  **represents** every **decidable** relation, it must be inconsistent.

**Decidable** properties (one-place relations) are represented by **formulas** with one free variable. Let  $\varphi_0(x), \varphi_1(x), \dots$ , be a computable enumeration of all such **formulas**. Now consider the following set  $D \subseteq \mathbb{N}$ :

$$D = \{n : \Gamma \vdash \neg\varphi_n(\bar{n})\}$$

The set  $D$  is **decidable**, since we can test if  $n \in D$  by first computing  $\varphi_n(x)$ , and from this  $\neg\varphi_n(\bar{n})$ . Obviously, substituting the term  $\bar{n}$  for every free occurrence of  $x$  in  $\varphi_n(x)$  and prefixing  $\varphi(\bar{n})$  by  $\neg$  is a mechanical matter. By assumption,  $\Gamma$  is **decidable**, so we can test if  $\neg\varphi(\bar{n}) \in \Gamma$ . If it is,  $n \in D$ , and if it isn't,  $n \notin D$ . So  $D$  is likewise **decidable**.

Since  $\Gamma$  **represents** all **decidable** properties, it **represents**  $D$ . And the **formulas** which **represent**  $D$  in  $\Gamma$  are all among  $\varphi_0(x), \varphi_1(x), \dots$ . So let  $d$  be a number such that  $\varphi_d(x)$  **represents**  $D$  in  $\Gamma$ . If  $d \notin D$ , then, since  $\varphi_d(x)$  **represents**  $D$ ,  $\Gamma \vdash \neg\varphi_d(\bar{d})$ . But that means that  $d$  meets the defining condition of  $D$ , and so  $d \in D$ . This contradicts  $d \notin D$ . So by indirect proof,  $d \in D$ .

Since  $d \in D$ , by the definition of  $D$ ,  $\Gamma \vdash \neg\varphi_d(\bar{d})$ . On the other hand, since  $\varphi_d(x)$  **represents**  $D$  in  $\Gamma$ ,  $\Gamma \vdash \varphi_d(\bar{d})$ . Hence,  $\Gamma$  is inconsistent.  $\square$

The preceding theorem shows that no theory that **represents** all **decidable** relations can be **decidable**. We will show that **Q** does **represent** all **decidable** relations; this means that all theories that include **Q**, such as **PA** and **TA**, also do, and hence also are not **decidable**. explanation

We can also use this result to obtain a weak version of the first incompleteness theorem. Any theory that is **axiomatizable** and **complete** is **decidable**. Consistent theories that are **axiomatizable** and **represent** all **decidable** properties then cannot be **complete**.

**Theorem int.2.** *If  $\Gamma$  is **axiomatizable** and **complete** it is **decidable**.*

*Proof.* Any inconsistent theory is **decidable**, since inconsistent theories contain all **sentences**, so the answer to the question “is  $\varphi \in \Gamma$ ” is always “yes,” i.e., can be decided.

So suppose  $\Gamma$  is consistent, and furthermore is **axiomatizable**, and **complete**. Since  $\Gamma$  is **axiomatizable**, it is **computably enumerable**. For we can enumerate

all the correct **derivations** from the axioms of  $\Gamma$  by a computable function. From a correct **derivation** we can compute the **sentence** it **derives**, and so together there is a computable function that enumerates all theorems of  $\Gamma$ . A **sentence** is a theorem of  $\Gamma$  iff  $\neg\varphi$  is not a theorem, since  $\Gamma$  is consistent and **complete**. We can therefore decide if  $\varphi \in \Gamma$  as follows. Enumerate all theorems of  $\Gamma$ . When  $\varphi$  appears on this list, we know that  $\Gamma \vdash \varphi$ . When  $\neg\varphi$  appears on this list, we know that  $\Gamma \not\vdash \varphi$ . Since  $\Gamma$  is **complete**, one of these cases eventually obtains, so the procedure eventually produces an answer.  $\square$

**Corollary int.3.** *If  $\Gamma$  is consistent, axiomatizable, and represents every decidable property, it is not complete.*

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*Proof.* If  $\Gamma$  were **complete**, it would be **decidable** by the previous theorem (since it is **axiomatizable** and consistent). But since  $\Gamma$  **represents** every **decidable** property, it is not **decidable**, by the first theorem.  $\square$

**Problem int.1.** Show that  $\mathbf{TA} = \{\varphi : \mathfrak{N} \models \varphi\}$  is not **axiomatizable**. You may assume that  $\mathbf{TA}$  represents all decidable properties.

Once we have established that, e.g.,  $\mathbf{Q}$ , **represents** all **decidable** properties, the corollary tells us that  $\mathbf{Q}$  must be incomplete. However, its proof does not provide an example of an independent **sentence**; it merely shows that such a **sentence** must exist. For this, we have to arithmetize syntax and follow Gödel's original proof idea. And of course, we still have to show the first claim, namely that  $\mathbf{Q}$  does, in fact, **represent** all **decidable** properties.

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## Bibliography