

int.1 Undecidability and Incompleteness

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Gödel's proof of the incompleteness theorems require arithmetization of syntax. But even without that we can obtain some nice results just on the assumption that a theory **represents** all **decidable** relations. The proof is a diagonal argument similar to the proof of the undecidability of the halting problem.

Theorem int.1. *If Γ is a consistent theory that **represents** every **decidable** relation, then Γ is not **decidable**.*

Proof. Suppose Γ were **decidable**. We show that if Γ **represents** every **decidable** relation, it must be inconsistent.

Decidable properties (one-place relations) are represented by **formulas** with one free variable. Let $\varphi_0(x), \varphi_1(x), \dots$, be a computable enumeration of all such **formulas**. Now consider the following set $D \subseteq \mathbb{N}$:

$$D = \{n : \Gamma \vdash \neg\varphi_n(\bar{n})\}$$

The set D is **decidable**, since we can test if $n \in D$ by first computing $\varphi_n(x)$, and from this $\neg\varphi_n(\bar{n})$. Obviously, substituting the term \bar{n} for every free occurrence of x in $\varphi_n(x)$ and prefixing $\varphi(\bar{n})$ by \neg is a mechanical matter. By assumption, Γ is **decidable**, so we can test if $\neg\varphi(\bar{n}) \in \Gamma$. If it is, $n \in D$, and if it isn't, $n \notin D$. So D is likewise **decidable**.

Since Γ **represents** all **decidable** properties, it **represents** D . And the **formulas** which **represent** D in Γ are all among $\varphi_0(x), \varphi_1(x), \dots$. So let d be a number such that $\varphi_d(x)$ **represents** D in Γ . If $d \notin D$, then, since $\varphi_d(x)$ **represents** D , $\Gamma \vdash \neg\varphi_d(\bar{d})$. But that means that d meets the defining condition of D , and so $d \in D$. This contradicts $d \notin D$. So by indirect proof, $d \in D$.

Since $d \in D$, by the definition of D , $\Gamma \vdash \neg\varphi_d(\bar{d})$. On the other hand, since $\varphi_d(x)$ **represents** D in Γ , $\Gamma \vdash \varphi_d(\bar{d})$. Hence, Γ is inconsistent. \square

The preceding theorem shows that no theory that **represents** all **decidable** relations can be **decidable**. We will show that **Q** does **represent** all **decidable** relations; this means that all theories that include **Q**, such as **PA** and **TA**, also do, and hence also are not **decidable**. explanation

We can also use this result to obtain a weak version of the first incompleteness theorem. Any theory that is **axiomatizable** and **complete** is **decidable**. Consistent theories that are **axiomatizable** and **represent** all **decidable** properties then cannot be **complete**.

Theorem int.2. *If Γ is **axiomatizable** and **complete** it is **decidable**.*

Proof. Any inconsistent theory is **decidable**, since inconsistent theories contain all **sentences**, so the answer to the question “is $\varphi \in \Gamma$ ” is always “yes,” i.e., can be decided.

So suppose Γ is consistent, and furthermore is **axiomatizable**, and **complete**. Since Γ is **axiomatizable**, it is **computably enumerable**. For we can enumerate

all the correct **derivations** from the axioms of Γ by a computable function. From a correct **derivation** we can compute the **sentence** it **derives**, and so together there is a computable function that enumerates all theorems of Γ . A **sentence** is a theorem of Γ iff $\neg\varphi$ is not a theorem, since Γ is consistent and **complete**. We can therefore decide if $\varphi \in \Gamma$ as follows. Enumerate all theorems of Γ . When φ appears on this list, we know that $\Gamma \vdash \varphi$. When $\neg\varphi$ appears on this list, we know that $\Gamma \not\vdash \varphi$. Since Γ is **complete**, one of these cases eventually obtains, so the procedure eventually produces an answer. \square

Corollary int.3. *If Γ is consistent, axiomatizable, and represents every decidable property, it is not complete.*

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Proof. If Γ were **complete**, it would be **decidable** by the previous theorem (since it is **axiomatizable** and consistent). But since Γ **represents** every **decidable** property, it is not **decidable**, by the first theorem. \square

Problem int.1. Show that $\mathbf{TA} = \{\varphi : \mathfrak{N} \models \varphi\}$ is not **axiomatizable**. You may assume that \mathbf{TA} represents all decidable properties.

Once we have established that, e.g., \mathbf{Q} , **represents** all **decidable** properties, the corollary tells us that \mathbf{Q} must be incomplete. However, its proof does not provide an example of an independent **sentence**; it merely shows that such a **sentence** must exist. For this, we have to arithmetize syntax and follow Gödel's original proof idea. And of course, we still have to show the first claim, namely that \mathbf{Q} does, in fact, **represent** all **decidable** properties.

It should be noted that not every *interesting* theory is incomplete or undecidable. There are many theories that are sufficiently strong to describe interesting mathematical facts that do not satisfy the conditions of Gödel's result. For instance, $\mathbf{Pres} = \{\varphi \in \mathcal{L}_{A^+} : \mathbb{N} \models \varphi\}$, the set of **sentences** of the language of arithmetic without \times true in the standard model, is both complete and decidable. This theory is called Presburger arithmetic, and proves all the truths about natural numbers that can be formulated just with 0 , $!$, and $+$.

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Bibliography