

int.1 Undecidability and Incompleteness

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Gödel's proof of the incompleteness theorems require arithmetization of syntax. But even without that we can obtain some nice results just on the assumption that a theory **represents** all **decidable** relations. The proof is a diagonal argument similar to the proof of the undecidability of the halting problem.

Theorem int.1. *If Γ is a consistent theory that **represents** every **decidable** relation, then Γ is not **decidable**.*

Proof. Suppose Γ were **decidable**. We show that if Γ **represents** every **decidable** relation, it must be inconsistent.

Decidable properties (one-place relations) are represented by **formulas** with one free variable. Let $\varphi_0(x), \varphi_1(x), \dots$, be a computable enumeration of all such **formulas**. Now consider the following set $D \subseteq \mathbb{N}$:

$$D = \{n : \Gamma \vdash \neg\varphi_n(\bar{n})\}$$

The set D is **decidable**, since we can test if $n \in D$ by first computing $\varphi_n(x)$, and from this $\neg\varphi_n(\bar{n})$. Obviously, substituting the term \bar{n} for every free occurrence of x in $\varphi_n(x)$ and prefixing $\varphi(\bar{n})$ by \neg is a mechanical matter. By assumption, Γ is **decidable**, so we can test if $\neg\varphi(\bar{n}) \in \Gamma$. If it is, $n \in D$, and if it isn't, $n \notin D$. So D is likewise **decidable**.

Since Γ **represents** all **decidable** properties, it **represents** D . And the **formulas** which **represent** D in Γ are all among $\varphi_0(x), \varphi_1(x), \dots$. So let d be a number such that $\varphi_d(x)$ **represents** D in Γ . If $d \notin D$, then, since $\varphi_d(x)$ **represents** D , $\Gamma \vdash \neg\varphi_d(\bar{d})$. But that means that d meets the defining condition of D , and so $d \in D$. This contradicts $d \notin D$. So by indirect proof, $d \in D$.

Since $d \in D$, by the definition of D , $\Gamma \vdash \neg\varphi_d(\bar{d})$. On the other hand, since $\varphi_d(x)$ **represents** D in Γ , $\Gamma \vdash \varphi_d(\bar{d})$. Hence, Γ is inconsistent. \square

The preceding theorem shows that no theory that **represents** all **decidable** relations can be **decidable**. We will show that **Q** does **represent** all **decidable** relations; this means that all theories that include **Q**, such as **PA** and **TA**, also do, and hence also are not **decidable**. explanation

We can also use this result to obtain a weak version of the first incompleteness theorem. Any theory that is **axiomatizable** and **complete** is **decidable**. Consistent theories that are **axiomatizable** and **represent** all **decidable** properties then cannot be **complete**.

Theorem int.2. *If Γ is **axiomatizable** and **complete** it is **decidable**.*

Proof. Any inconsistent theory is **decidable**, since inconsistent theories contain all **sentences**, so the answer to the question “is $\varphi \in \Gamma$ ” is always “yes,” i.e., can be decided.

So suppose Γ is consistent, and furthermore is **axiomatizable**, and **complete**. Since Γ is **axiomatizable**, it is **computably enumerable**. For we can enumerate

all the correct **derivations** from the axioms of Γ by a computable function. From a correct **derivation** we can compute the **sentence** it **derives**, and so together there is a computable function that enumerates all theorems of Γ . A **sentence** is a theorem of Γ iff $\neg\varphi$ is not a theorem, since Γ is consistent and **complete**. We can therefore decide if $\varphi \in \Gamma$ as follows. Enumerate all theorems of Γ . When φ appears on this list, we know that $\Gamma \vdash \varphi$. When $\neg\varphi$ appears on this list, we know that $\Gamma \not\vdash \varphi$. Since Γ is **complete**, one of these cases eventually obtains, so the procedure eventually produces an answer. \square

Corollary int.3. *If Γ is consistent, axiomatizable, and represents every decidable property, it is not complete.*

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Proof. If Γ were **complete**, it would be **decidable** by the previous theorem (since it is **axiomatizable** and consistent). But since Γ **represents** every **decidable** property, it is not **decidable**, by the first theorem. \square

Problem int.1. Show that $\mathbf{TA} = \{\varphi : \mathfrak{N} \models \varphi\}$ is not **axiomatizable**. You may assume that \mathbf{TA} represents all decidable properties.

Once we have established that, e.g., \mathbf{Q} , **represents** all **decidable** properties, the corollary tells us that \mathbf{Q} must be incomplete. However, its proof does not provide an example of an independent **sentence**; it merely shows that such a **sentence** must exist. For this, we have to arithmetize syntax and follow Gödel's original proof idea. And of course, we still have to show the first claim, namely that \mathbf{Q} does, in fact, **represent** all **decidable** properties.

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Bibliography