int.1 Undecidability and Incompleteness

inc:int:dec:sec Gödel's proof of the incompleteness theorems require arithmetization of syntax. But even without that we can obtain some nice results just on the assumtion that a theory represents all decidable relations. The proof is a diagonal argument similar to the proof of the undecidability of the halting problem.

Theorem int.1. *If Γ is a consistent theory that represents every decidable relation, then Γ is not decidable.*

Proof. Suppose Γ were decidable. We show that if Γ represents every decidable relation, it must be inconsistent.

Decidable properties (one-place relations) are represented by formulas with one free variable. Let $\varphi_0(x)$, $\varphi_1(x)$, …, be a computable enumeration of all such formulas. Now consider the following set $D \subseteq \mathbb{N}$:

$$D = \{n : \Gamma \vdash \lnot\varphi_n(\overline{n})\}$$

The set D is decidable, since we can test if $n \in D$ by first computing $\varphi_n(x)$, and from this $\lnot\varphi_n(\overline{n})$. Obviously, substituting the term $\overline{n}$ for every free occurrence of x in $\varphi_n(x)$ and prefixing $\varphi(\overline{n})$ by $\lnot$ is a mechanical matter. By assumption, Γ is decidable, so we can test if $\lnot\varphi(\overline{n}) \in \Gamma$. If it is, $n \in D$, and if it isn't, $n \notin D$. So D is likewise decidable.

Since Γ represents all decidable properties, it represents D. And the formulas which represent D in Γ are all among $\varphi_0(x)$, $\varphi_1(x)$, …. So let d be a number such that $\varphi_d(x)$ represents D in Γ. If $d \notin D$, then, since $\varphi_d(x)$ represents D, $\Gamma \vdash \lnot\varphi_d(\overline{d})$. But that means that d meets the defining condition of D, and so $d \in D$. This contradicts $d \notin D$. So by indirect proof, $d \in D$.

Since $d \in D$, by the definition of D, $\Gamma \vdash \lnot\varphi_d(\overline{d})$. On the other hand, since $\varphi_d(x)$ represents D in Γ, $\Gamma \vdash \varphi_d(\overline{d})$. Hence, Γ is inconsistent. □

The preceding theorem shows that no theory that represents all decidable relations can be decidable. We will show that **Q** does represent all decidable relations; this means that all theories that include **Q**, such as **PA** and **TA**, also do, and hence also are not decidable. explanation

We can also use this result to obtain a weak version of the first incompleteness theorem. Any theory that is axiomatizable and complete is decidable. Consistent theories that are axiomatizable and represent all decidable properties then cannot be complete.

Theorem int.2. *If Γ is axiomatizable and complete it is decidable.*

Proof. Any inconsistent theory is decidable, since inconsistent theories contain all sentences, so the answer to the question "is $\varphi \in \Gamma$" is always "yes," i.e., can be decided.

So suppose Γ is consistent, and furthermore is axiomatizable, and complete. Since Γ is axiomatizable, it is computably enumerable. For we can enumerate all the correct derivations from the axioms of Γ by a computable function. From a correct derivation we can compute the sentence it derives, and so together there is a computable function that enumerates all theorems of Γ. A sentence is a theorem of Γ iff $\neg\varphi$ is not a theorem, since Γ is consistent and complete. We can therefore decide if $\varphi \in \Gamma$ as follows. Enumerate all theorems of Γ. When φ appears on this list, we know that $\Gamma \vdash \varphi$. When $\neg\varphi$ appears on this list, we know that $\Gamma \nvdash \varphi$. Since Γ is complete, one of these cases eventually obtains, so the procedure eventually produces and answer. □

Corollary int.3. *If Γ is consistent, axiomatizable, and represents every decidable property, it is not complete.* *inc:int:dec:cor:incompleteness*

Proof. If Γ were complete, it would be decidable by the previous theorem (since it is axiomatizable and consistent). But since Γ represents every decidable property, it is not decidable, by the first theorem. □

Problem int.1. Show that $\mathbf{TA} = \{\varphi : \mathfrak{N} \models \varphi\}$ is not axiomatizable. You may assume that $\mathbf{TA}$ represents all decidable properties.

Once we have established that, e.g., $\mathbf{Q}$, represents all decidable properties, the corollary tells us that $\mathbf{Q}$ must be incomplete. However, its proof does not provide an example of an independent sentence; it merely shows that such a sentence must exist. For this, we have to arithmetize syntax and follow Gödel's original proof idea. And of course, we still have to show the first claim, namely that $\mathbf{Q}$ does, in fact, represent all decidable properties.

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Bibliography