int.1 Overview of Incompleteness Results

Hilbert expected that mathematics could be formalized in an axiomatizable theory which it would be possible to prove complete and decidable. Moreover, he aimed to prove the consistency of this theory with very weak, "finitary," means, which would defend classical mathematics agianst the challenges of intuitionism. Gödel's incompleteness theorems showed that these goals cannot be achieved.

Gödel's first incompleteness theorem showed that a version of Russell and Whitehead's *Principia Mathematica* is not complete. But the proof was actually very general and applies to a wide variety of theories. This means that it wasn't just that *Principia Mathematica* did not manage to completely capture mathematics, but that *no* acceptable theory does. It took a while to isolate the features of theories that suffice for the incompleteness theorems to apply, and to generalize Gödel's proof to apply make it depend only on these features. But we are now in a position to state a very general version of the first incompleteness theorem for theories in the language $\mathcal{L}_A$ of arithmetic.

Theorem int.1. *If Γ is a consistent and axiomatizable theory in $\mathcal{L}_A$ which represents all computable functions and decidable relations, then Γ is not complete.*

To say that Γ is not complete is to say that for at least one sentence φ, $\Gamma \nvdash \varphi$ and $\Gamma \nvdash \neg\varphi$. Such a sentence is called *independent* (of Γ). We can in fact relatively quickly prove that there must be independent sentences. But the power of Gödel's proof of the theorem lies in the fact that it exhibits a *specific example* of such an independent sentence. The intriguing construction produces a sentence G_Γ, called a *Gödel sentence* for Γ, which is unprovable because in Γ, G_Γ is equivalent to the claim that G_Γ is unprovable in Γ. It does so *constructively*, i.e., given an axiomatization of Γ and a description of the proof system, the proof gives a method for actually writing down G_Γ.

The construction in Gödel's proof requires that we find a way to express in $\mathcal{L}_A$ the properties of and operations on terms and formulas of $\mathcal{L}_A$ itself. These include properties such as "φ is a sentence," "δ is a derivation of φ," and operations such as $\varphi[t/x]$. This way must (a) express these properties and relations via a "coding" of symbols and sequences thereof (which is what terms, formulas, derivations, etc. are) as natural numbers (which is what $\mathcal{L}_A$ can talk about). It must (b) do this in such a way that Γ will prove the relevant facts, so we must show that these properties are coded by decidable properties of natural numbers and the operations correspond to computable functions on natural numbers. This is called "arithmetization of syntax."

Before we investigate how syntax can be arithmetized, however, we will consider the condition that Γ is "strong enough," i.e., represents all computable functions and decidable relations. This requires that we give a precise definition of "computable." This can be done in a number of ways, e.g., via the model of Turing machines, or as those functions computable by programs in some general-purpose programming language. Since our aim is to represent these

functions and relations in a theory in the language $\mathcal{L}_A$, however, it is best to pick a simple definition of computability of just numerical functions. This is the notion of *recursive function*. So we will first discuss the recursive functions. We will then show that **Q** already represents all recursive functions and relations. This will allow us to apply the incompleteness theorem to specific theories such as **Q** and **PA**, since we will have established that these are examples of theories that are "strong enough."

The end result of the arithmetization of syntax is a formula $\mathsf{Prov}_\Gamma(x)$ which, via the coding of formulas as numbers, expresses provability from the axioms of Γ. Specifically, if φ is coded by the number n, and $\Gamma \vdash \varphi$, then $\Gamma \vdash \mathrm{Prov}_\Gamma(\overline{n})$. This "provability predicate" for Γ allows us also to express, in a certain sense, the consistency of Γ as a sentence of $\mathcal{L}_A$: let the "consistency statemetn" for Γ be the sentence $\neg\mathrm{Prov}_\Gamma(\overline{n})$, where we take n to be the code of a contradiction, e.g., of $\bot$. The second incompleteness theorem states that consistent axiomatizable theories also do not prove their own consistency statements. The conditions required for this theorem to apply are a bit more stringent than just that the theory represents all computable functions and decidable relations, but we will show that **PA** satisifes them.

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Bibliography